Mathematics Spectrum

FEBRUARY 2016 ₹35

JEE ADVANCED
PERMUTATION AND COMBINATION
From The Editor's Desk
MASTERING MATHEMATICS

JEE
Final Touch
Fast Track Revision

BINOMIAL THEOREM AND MATHEMATICAL INDUCTION
PERMUTATION & COMBINATION
PROBABILITY
FROM CLASS 11 SYLLABUS

INDEFINITE INTEGRALS
DEFINITE INTEGRALS
APPLICATION OF DERIVATIVE
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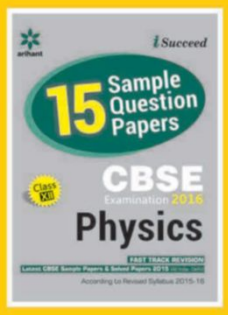

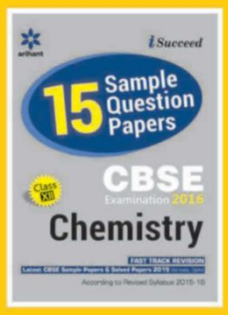

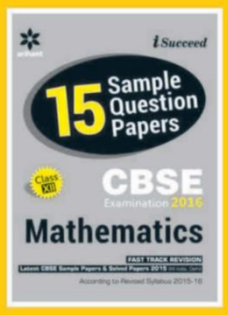

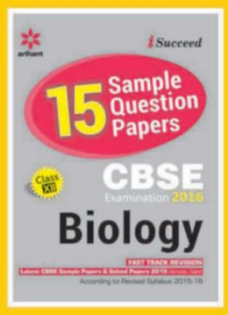

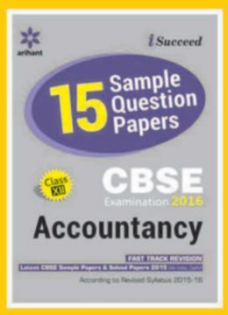

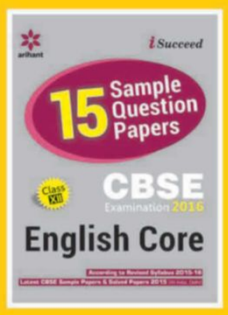

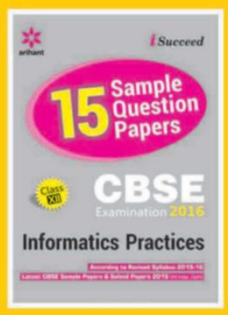

Only Sample Question Papers According to the Latest CBSE Pattern

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FROM THE EDITOR'S DESK

MASTERING MATHEMATICS

"Mathematics is a great motivator for all humans because its career starts with 'zero', but it never ends (Infinity)".

Mathematics is a subject that has many long range objectives. It enjoys special esteem above all other Science because its laws are absolute and indisputable, while the laws of other Science are up to a certain extent debatable and always in a danger of being overthrown by newly discovered facts.

This subject is around us since the beginning of time till today. It lies in nature to all real applications of our daily life from-

- Numbering any fruit to how much water we add to cook rice?
- Or by how many inches you have grown in last few years?
- Or how much amount is left in bank?

All questions can be answered through Mathematics only.

Many early civilisations including the Egyptians studied numbers as a Science and recorded their discoveries. In India, Maths was largely centered on problems in astronomy. However, India is in particular noted for developing the concept of zero that was passed into Western Mathematics through Arabic tradition and is as important as a placeholder in our modern decimal system.

Unlike other subjects, Mathematics, has more to do with skills than knowledge and content. In real sense, it (Mathematics) is all about mastering a collection of skills and their application to solve problems. **The functioning of Mathematics involves understanding a concept which is followed by practice of its various applications through problems.** This subject does not require an ability to memorise, but instead extreme devotion is required towards it. A daily practice routine is essential to expertise in this subject. In all, Mathematics is an exciting subject, one of the greatest achievements of human intellect. My personal feelings about this subject are not different from others. In my opinion this subject is not only useful, but is intrinsically beautiful too.

MANTRAS TO CAPTURE MATHEMATICS BETTER

Mathematics is not like other subjects. It requires more to do skills than knowledge and content. In other words we can say that the capability to apply your knowledge and repeated practice are the two major demands of this subject. *Some general mantras to grip this subject are given below*

HOMEWORK AND SELF STUDY BOTH ARE IMPORTANT

Homework is a 'practice' work set given by your teacher each day to reinforce your learning of a particular skill while self study on the other hand is your own revision work for strengthening and improving your understanding of the subject. Thus, especially for this subject the prime requirement is to work separately on both of these aspects.

LEARNING BY HALVES IS IMPOSSIBLE IN MATHS

Unlike other subjects, complete mathematical knowledge is interconnected, i.e. the theory and skills of one mathematical topic are connected with other. This sequential nature of Maths show gradual increase in complexity also. So, here the learning must be done in a particular order, so as to avoid any miss out which in turn makes it more difficult to progress.

MATHEMATICAL PROBLEMS ARE PREDICTABLE

In comparison to other subjects, the types of questions asked in Mathematics are more predictable and conventional. Guessing such questions simply depends upon how you have mastered your mathematical skills.

PRACTICE IS THE KEY

Regular practice is the only way to be a master of calculations. The practice will not only help you to understand and remember the work, but also increases your familiarity with the subject. It is best to begun with easier examples while practicing.

AGGRESSIVENESS PAYS A LOT

For a mathematician being aggressive means more attacking towards his weak points. So, it is suggestive don't give up soon, be patient in your approach and use following tips to get good hold over your Mathematics -

- Try to minimise the use of hints, calculators etc.
- Challenge yourself and remain enthusiastic.
- Do not hesitate to start again, if you are not sure of procedure/solution.

TESTING HELPS IN FILTERING WRONG AND RIGHT

Testing with the help of revision exercises, assignment and past exams is a must to filter your weak and strong areas. Besides exams also helps in the development and improvement of exam temperament in you.

STRATEGISE OR PLAN UP IN ACCORDANCE WITH YOUR STRENGTHS AND WEAKNESS

Some students cry over the difficulties they face in indefinite integrals, while some others go beyond the limits in practicing the questions of this segment. At this juncture, be professional, "you are not doing Ph.D on indefinite integrals" infact that is not the requirement of IIT-JEE which contains a max. 2 to 3 questions on this segment.

Likewise I saw many students enthusiastically working on the problems of a particular section and neglecting others. Although, they work very hard in doing so. But if the requirement is not of so much than why to go for such a hard work. That's why I always prefer to make a strategy or planning keeping target in mind. This will always save you from overdoing one part at the cost of the other.

Remember! One more thing

"Life is a test if you woke up alive this morning,
it means you got an 'A' for yesterday"

RON FRY

@CLASS XI SYLLABUS

JEE Final Touch

Fast Track Revision

Binomial Theorem and Mathematical Induction

Binomial theorem is one of the important part of algebra. By the analysis of last three years paper of JEE, it is found that every year atleast one question is generally asked. Here, we have split the whole chapter into two topics along with practice questions which will help you to understand the concept of this chapter in easy way.

BINOMIAL THEOREM (FOR POSITIVE INTEGRAL INDEX)

Binomial Theorem Statement

If n is a positive integer, then the expansion of $(x + a)^n$ can be written as

$$(x + a)^n = {}^nC_0 x^n a^0 + {}^nC_1 x^{n-1} a + {}^nC_2 x^{n-2} a^2 + \ldots + {}^nC_r x^{n-r} a^r + \ldots {}^nC_n x^0 a^n$$

Observations

(i) This expansion has $(n + 1)$ terms.

(ii) In each term sum of powers of 'x' and 'a' is always equal to n.

(iii) ${}^nC_0, {}^nC_1, {}^nC_2, \ldots, {}^nC_r, \ldots, {}^nC_n$ are called binomial coefficient.

(iv) Binomial coefficient of the terms equidistant from beginning and end are same, i.e. ${}^nC_r = {}^nC_{n-r}$.

(v) pth term from the end is $(n - p + 2)$th term from the beginning.

(vi) $S = \sum_{r=0}^{n} {}^nC_r x^{n-r} a^r = \sum_{r=0}^{n} {}^nC_{n-r} x^r a^{n-r}$

(vii) ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$

(viii) $r \times {}^nC_r = n \times {}^{n-1}C_{r-1}$

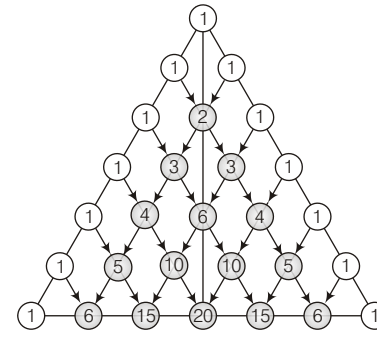
(ix) $\dfrac{{}^nC_r}{{}^nC_{r-1}} = \dfrac{n - r + 1}{r}$

(x) If ${}^nC_x = {}^nC_y \Rightarrow$ either $x = y$ or $x + y = n$

So, ${}^nC_r = {}^nC_{n-r} = \dfrac{n!}{r!\,n - r!}$

Pascal's Triangle

The coefficients of binomial expansion follow a particular pattern represented below



In the above triangle, any entry except the first and the last in the row is the sum of two entries above it in the preceding row. The above pattern is called Pascal's triangle.

SOME STANDARD EXPANSIONS

(i) $(x - a)^n = {}^nC_0 x^n a^0 - {}^nC_1 x^{n-1} a^1 + {}^nC_2 x^{n-2} a^2 - {}^nC_3 x^{n-3} a^3 + \ldots + (-1)^n {}^nC_n x^0 a^n$

i.e. $(x - a)^n = \sum_{r=0}^{n} (-1)^r {}^nC_r x^{n-r} a^r$

(ii) $(1+x)^n = {}^nC_0 x^0 + {}^nC_1 x + {}^nC_2 x^2 + {}^nC_3 x^3 + \ldots + {}^nC_n x^n$,

i.e. $(1+x)^n = \sum_{0}^{n} {}^nC_r x^r$

(iii) $(1-x)^n = {}^nC_0 x^0 - {}^nC_1 x + {}^nC_2 x^2 - \ldots + (-1)^n {}^nC_n x^n$

i.e. $(1-x)^n = \sum_{r=0}^{n} (-1)^r {}^nC_r x^r$

(iv) $(x+a)^n + (x-a)^n = 2[{}^nC_0 x^n a^0 + {}^nC_2 x^{n-2} a^2 + \ldots]$

(v) $(x+a)^n - (x-a)^n = 2[{}^nC_1 x^{n-1} a + {}^nC_3 x^{n-3} a^3 + \ldots]$

Tips to Find Coefficient of Required Terms

- Coefficient of x^r in expansion of $(1+x)^n$ is ${}^nC_r$.
- Coefficient of x^r in expansion of $(1+ax)^n$ is ${}^nC_r a^r$.
- Coefficient of x^r in expansion of $(a+bx)^n$ is ${}^nC_r b^r a^{n-r}$.
- Coefficient of x^r in expansion of $(1+x^p)^n$ is ${}^nC_{r/p}$, if r is a multiple of p.

General Term

In the expansion of $(x+a)^n$, general term is $(r+1)$th term which can be represented as T_{r+1}.

$T_{r+1} = {}^nC_r\, x^{n-r} a^r = {}^nC_r$ (First term)$^{n-r}$ (Second term)^r

Note Independent term or constant term of binomial expansion is the term in which exponent of variable is zero.

To find the term independent of x in the expansion of $(x+a)^n$ *we use following steps.*

Step I Write down the general term in the expansion of $(x+a)^n$ i.e. $(r+1)$th term $\times T_{r+1} = {}^nC_r\, x^{n-r} a^r$

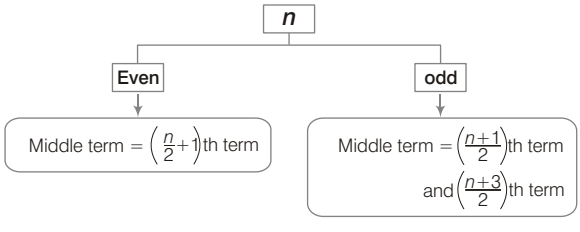
Step II Separate constant and variable by grouping.

Step III Put the index of x obtained in step II equal to zero and simplify for 'r'.

We will get the value of r for which there exists a term independent of x in the expansion.

Middle Term

The middle term depends on the value of n (even or odd). The following diagram will help you to find the middle term in the expansion of $(x+a)^n$.



Note

(i) If there are two middle terms in the expansion, then their binomial coefficients are equal.

(ii) Binomial coefficient of middle term is the greatest binomial coefficient.

Greatest Term

To find the greatest term in the expansion of $(1+x)^n$ follow the steps given below.

Step I Find the value of $\frac{x(n+1)}{x+1}$. Let $\frac{x(n+1)}{x+1} = m$

Step II

Case I If m is integer

Then, T_m or T_{m+1} are numerically equal and both are greatest term.

Case II If m is not integer

Then, $T_{[m]+1}$ is greatest term, where $[\times]$ denotes the greatest integer function.

Method to Find Remainder Using Binomial Theorem

To find the remainder, when a^n is divided by b, we adjust the power of a to a^k, which is very close to b or multiple of b, if possible with difference 1. Also, it is to be noted down the remainder is always positive, e.g. When 5^{99} is divided by 13.

$N = 5^{99} = 5(5^{98}) = 5(5^2)^{49} = 5(26-1)^{49}$

$\Rightarrow N = 5[{}^{49}C_0 26^{49} - {}^{49}C_1 26^{48} \ldots - {}^{49}C_{48} 26 + {}^{49}C_{49}]$

$\Rightarrow N = 5[{}^{49}C_0 26^{49} - {}^{49}C_1 26^{48} + \ldots + {}^{49}C_{48} 26] - 5$

$\Rightarrow \frac{N}{13} = \frac{5}{13}[{}^{49}C_0 26^{49} - {}^{49}C_1 26^{48} + \ldots + {}^{49}C_{48} 26] - \frac{5}{13}$

$\Rightarrow \frac{N}{13} = \frac{5}{13}[{}^{49}C_0 26^{49} - {}^{49}C_1 26^{48} + \ldots + {}^{49}C_{48} 26] - 1 + \frac{8}{13}$

[here, 1 is added or subtracted to make remainder positive]

Remainder = 8

Properties of Binomial Coefficient

Following are important properties of binomial coefficients. These results will help you in great extent to minimise the problem solving time.

(i) If n is even, then greatest coefficient is ${}^nC_{n/2}$.

(ii) If n is odd, then greatest coefficient is ${}^nC_{\frac{n-1}{2}}$ or ${}^nC_{\frac{n+1}{2}}$

(iii) $C_0 + C_1 + C_2 + \ldots + C_n = 2^n$

(iv) $C_0 - C_1 + C_2 - C_3 + \ldots + (-1)^n C_n = 0$

(v) $C_0 + C_2 + C_4 + \ldots = C_1 + C_3 + C_5 + \ldots = 2^{n-1}$

(vi) $C_0^2 + C_2^2 + C_4^2 + \ldots + C_n^2 = \frac{(2n)!}{n!n!} = {}^{2n}C_n$

(vii) $C_0^2 - C_1^2 + C_2^2 - C_3^2 + \ldots = \begin{cases} 0, & \text{if } n \text{ is odd} \\ (-1)^{n/2} \times C_{n/2}, & \text{if } n \text{ is even} \end{cases}$

(viii) $C_0C_1 + C_1C_2 + C_2C_3 + \ldots + C_{n-1}C_n = {}^{2n}C_{n+1}$

(ix) $C_0C_r + C_1C_{r+1} + \ldots + C_{n-r}C_n = {}^{2n}C_{n-r}$ or ${}^{2n}C_{n+r}$

(x) $1C_1 + 2C_2 + 3C_3 + \ldots + nC_n = n \times 2^{n-1}$

(xi) $C_1 - 2C_2 + 3C_3 - \ldots = 0$

(xii) $C_0 + 2C_1 + 3C_2 + \ldots + (n+1)C_n = (n+2)2^{n-1}$

(xiii) ${}^nC_0 + {}^{n+1}C_1 + {}^{n+2}C_2 + \ldots + {}^{n+n}C_n = {}^{2n+1}C_{n+1}$

(xiv) ${}^nC_0\,{}^{2n}C_n - {}^nC_1\,{}^{2n-1}C_n + {}^nC_2\,{}^{2n-2}C_n + \ldots + (-1)^n {}^nC_n\,{}^nC_n = 1$

Important Theorem

If $(\sqrt{A}+B)^n = I + f$, where I and n are positive integers, n being odd and $0 < f < 1$, then $(I+f)(I-f) = k^n$, where $A - B^2 = k > 0$ and $\sqrt{A} - B < 1$. If n is an even integer, then $(\sqrt{A}+B)^n + (\sqrt{A} - B)^n = I + f + f$ Hence, LHS and I are integers. Then, $f + f$ is also an integer $\Rightarrow f + f = 1$

$f(1 - f) \Rightarrow (\sqrt{A}+B)^n(\sqrt{A} - B)^n = (A - B^2)^n = k^n$

Sum of the Series, when *i* and *j* are Dependent

If sum of the series $\sum_{0 \le i<j \le n} f(i)f(j)$ is given, then i and j are not independent. In this summation, three types of terms occur, those when $i < j$, $i > j$ and $i = j$. Also, sum of terms, when $i < j$ is equal to sum of the terms when $i > j$, if $f(i)$ and $f(j)$ are symmetrical.

$$\sum_{i=1}^{n}\sum_{j=1}^{n} f(i)f(j) = \sum_{0 \le i<j \le n} f(i)f(j) + \sum_{0 \le i<j \le n} f(i)f(j) + \sum_{i=j} f(i)f(j)$$

$$\sum_{i=1}^{n}\sum_{j=1}^{n} f(i)f(j) = 2\sum_{0 \le i<j \le n} f(i)f(j) + \sum_{i=j} f(i)f(j)$$

Hence, $$\sum_{0 \le i<j \le n} f(i)f(j) = \frac{\sum_{i=1}^{n}\sum_{j=1}^{n} f(i)f(j) - \sum_{i=j} f(i)f(j)}{2}$$

Also, $$\sum_{0 \le i<j \le n} f(i)f(j) = \frac{\sum_{i=1}^{n}\sum_{j=1}^{n} f(i)f(j) + \sum_{i=j} f(i)f(j)}{2} \quad [\because 0 \le i \le j \le n]$$

Important Results

(i) $\sum_{0 \le i<j \le n} {}^nC_i \times {}^nC_j = \frac{2^{2n} - {}^{2n}C_n}{2}$

(ii) $\sum_{0 \le i \le j \le n} {}^nC_i\,{}^nC_j = \frac{2^{2n} + {}^{2n}C_n}{2}$

(iii) $\sum_{i}\sum_{j} {}^nC_i\,{}^nC_j = 2^{2n} - {}^{2n}C_n$

Note

(i) Here, $|x| < 1$ is necessary, if n is rational number other than whole number.

(ii) There are infinite number of terms in the expansion of $(1+x)^n$, when n is integer or fraction.

(iii) Make the first term unity, if it is other than unity, when n is integer or fractional

e.g. $(a+bx)^n = a^n\left(1 + \frac{bx}{a}\right)^n$, here $\frac{bx}{a} < 1$

(iv) The general term in the expansion of $(1+x)^n$ is given by

$$T_{r+1} = \frac{n(n-1)(n-2)\ldots\{n-(r-1)\}}{r!}x^r$$

(v) Expansion of

$$(1-x)^{-n} = 1 + nx + \frac{n(n+1)}{2!}x^2 + \frac{n(n+1)(n+2)}{3!}x^3 + \ldots + \frac{n(n+1)(n+2)\ldots(n+r-1)}{r!}x^r + \ldots$$

(vi) In the expansion of $(1-x)^{-n}$, coefficient of x^r is ${}^{n+r-1}C_r$.

Best Practice SHOTS

1. The number of terms which are free from radical sign in the expansion of $(y^{1/5} + x^{1/10})^{55}$ is

(a) 7 (b) 6
(c) 5 (d) None of these

2. If the second third and fourth terms in the expansion of $(x+y)^n$ are 135, 30 and $\frac{10}{3}$ respectively, then

(a) $n = 6$ (b) $n = 5$
(c) $n = 7$ (d) None of these

3. The greatest value of the term independent of x in the expansion of $(x\sin\theta + x^{-1}\cos\theta)^{10}$, $\theta \in R$ is

(a) 2^5 (b) $\frac{10!}{(5!)^2}$
(c) $\frac{10!}{2^5(5!)^2}$ (d) None of these

4. The coefficient of the middle term in the binomial expansion in powers of x, of $(1+x)^4$ and of $(1-x)^6$ is same if α equals **[AIEEE 2004]**

(a) $\frac{3}{5}$ (b) $\frac{10}{3}$ (c) $\frac{3}{10}$ (d) $\frac{5}{3}$

5. The number of $101^{100} - 1$ is divisible by

(a) 100 (b) 1000 (c) 10000 (d) All of these

6. P is a set containing n elements. A subset A of P is chosen and the set P is reconstructed by replacing the element A. A subset B of P is chosen again. The number of ways of choosing A and B such that A and B have equal number of elements, is

(a) 2^n (b) 3^n (c) $(2^n)^2$ (d) ${}^{2n}C_n$

7. In question 6, the number of ways of choosing A and B such that B is a subset of A, is

(a) 2^n (b) 3^n
(c) ${}^{2n}C_n$ (d) None of these

8. In question 6, the number of ways of choosing A and B such that A and B have no common element, is

(a) 2^n (b) 3^n
(c) 4^n (d) None of these

9. The value of $\sum_{r=0}^{n}\sum_{p=0}^{r} {}^nC_r \times {}^rC_p$ is equal to

(a) $3^n - 2^n$ (b) $3^n - 2^n - 2$
(c) $3^n - 2^n + 2$ (d) None of these

10. 7^{103} when divided by 25 leaves the remainder
(a) 20 (b) 16 (c) 18 (d) 15

11. If $(1+x)^n = C_0 + C_1x + C_2x^2 + \ldots + C_nx^n$, then the sum $C_0 + (C_0 + C_1) + \ldots + (C_0 + C_1 + \ldots + C_{n-1})$ is equal to
(a) $n \times 2^{n-1}$ (b) $(n-1)2^n$ (c) $n \times 2^n$ (d) $(n+1)2^n$

12. If $a_n = \sum_{r=0}^{n} \frac{1}{{}^nC_r}$, then $\sum_{r=0}^{n} \frac{r}{{}^nC_r}$ equals
(a) $(n-1)a_n$ (b) $n \times a_n$ (c) $\frac{n}{2}a_n$ (d) None

13. If O is the sum of odd terms and E that of even terms in the expansion of $(x+a)^n$, then $O^2 - E^2$ is equal to
(a) $(x^2 + a^2)^n$ (b) $(x^2 - a^2)^n$
(c) $(x-a)^{2n}$ (d) None of these

14. The coefficient of the term independent of x in the expansion of $(1 + x + 2x^3)\left(\frac{3}{2}x^2 - \frac{1}{3x}\right)^9$ is
(a) $\frac{1}{3}$ (b) $\frac{17}{54}$ (c) $\frac{1}{4}$ (d) $\frac{19}{54}$

15. The third term in the expansion of $\left(\frac{1}{x} + x^{\log_{10} x}\right)^5$, $x > 1$ is 1000, then x equals
(a) 100 (b) 10 (c) 1 (d) $\frac{1}{\sqrt{10}}$

16. If $R = (7 + 4\sqrt{3})^{2n} = I + f$, where $I \in N$ and $0 < f < 1$, then $R(1-f)$ equals
(a) $(7 - 4\sqrt{3})^{2n}$ (b) $\frac{1}{(7 + 4\sqrt{3})^{2n}}$
(c) 1 (d) None of these

17. If $R = (6\sqrt{6} + 14)^{2n+1}$ and $f = R - [R]$, where $[\times]$ denotes the greatest integer function, then Rf equals
(a) 20^n (b) 20^{2n}
(c) 20^{2n+1} (d) None of these

18. The greatest term in the expansion of $(1 + 3x)^{54}$, when $x = \frac{1}{3}$ is
(a) 28th (b) 25th
(c) 26th (d) 24th

19. The interval in which x must lie so that the greatest term in the expansion of $(1+x)^{2n}$ has the greatest coefficient, is
(a) $\left(\frac{n}{n+1}, \frac{n+1}{n}\right)$ (b) $\left(\frac{n}{n+1}, \frac{n+1}{n}\right)$
(c) $\left(\frac{n}{n+2}, \frac{n+2}{n}\right)$ (d) None of these

20. If ${}^nC_0, {}^nC_1, {}^nC_2, \ldots, {}^nC_n$ denote the binomial coefficient in the expansion of $(1+x)^n$ and $p + q = 1$, then $\sum_{r=0}^{n} r\,2^r\,{}^nC_r p^r q^{n-r}$ is equal to
(a) npq (b) $np(p+q)$ (c) $np(np+q)$ (d) $np(p+nq)$

21. If $C_0, C_1, C_2, \ldots, C_n$ denote the binomial coefficient in the expansion of $(1+x)^n$, then $\sum_{0}\sum_{r<s\le n} C_r C_s$ is equal to
(a) $2^{2n} - {}^{2n}C_n$ (b) ${}^{2n}C_n - 2^{2n}$
(c) $\frac{1}{2}(2^{2n} - {}^{2n}C_n)$ (d) $\frac{1}{2}(2^{2n} + {}^{2n}C_n)$

22. The sum of the coefficient of all the integral powers of x in the expansion of $(1 + 2\sqrt{x})^{40}$ is
(a) $3^{40} + 1$ (b) $3^{40} - 1$ (c) $\frac{1}{2}(3^{40} - 1)$ (d) $\frac{1}{2}(3^{40} + 1)$

23. The remainder when 9^{103} is divided by 25 is equal to
(a) 5 (b) 6
(c) 4 (d) None of these

24. If $f(x) = x^n$, then the value of
$$f(1) + \frac{f'(1)}{1!} + \frac{f''(1)}{2!} + \ldots + \frac{f^n(1)}{n!},$$
where $f^r(x)$ denotes the rth order derivative of $f(x)$ with respect to x, is
(a) n (b) 2^n
(c) 2^{n-1} (d) None of these

25. Let $r > 1$ and $n > 2$ be integers. Suppose, L and M are the coefficient of $(3r)$th and $(r+2)$th terms respectively in the binomial expansion of $2n(1+x)^{2n-1}$. If $(r+2)L = 3r(M)$, then n is **[AIEEE 2012]**
(a) $2r-1$ (b) $2r$ (c) $2r+1$ (d) $2r+2$

Self Study Guide for
Online BITSAT
2016

MULTINOMIAL THEOREM, BINOMIAL THEOREM FOR ANY INDEX

Multinomial Theorem

The expansion of $(x_1 + x_2 + x_3 + \ldots + x_r)^n$

$$= \frac{n!}{n_1!\,n_2!\,n_3!\ldots n_r!} x_1^{n_1} \times x_2^{n_2} \ldots x_r^{n_r},$$

where $n_1 + n_2 + n_3 + \ldots + n_r = n$ and $0 \quad n_1, n_2, \ldots, n_r \quad n$

IMPORTANT POINTS

(i) Coefficient of $x_1^{n_1}, x_2^{n_2}, \ldots, x_r^{n_r}$ in expansion of $(x_1 + x_2 + x_3 + \ldots + x_r)^n$ is $\frac{n!}{n_1!n_2!n_3!\ldots n_r!}$.

(ii) The greatest coefficient in expansion of $(x_1 + x_2 + x_3 + \ldots + x_m)^n$ is $\frac{n!}{q!^{m-r}[(q+1)!]^r}$.

where, q is quotient and r is remainder, when n is divided by m.

(iii) If n is positive integer and $a_1, a_2, \ldots, a_m \quad C$, then coefficient of x^r in the expansion of $(a_1 + a_2 x + a_3 x^2 + \ldots + a_{n\ 1} x^{m\ 1})^n$ is

$$\frac{n!}{n_1!\ n_2!\,n_3!\ldots n_m!}, a_1^{n_1} \times a_2^{n_2} \times \ldots, a_m^{n_m}$$

Here, $n_1 + n_2 + \ldots + n_m = n$

and $\quad n_2 + 2n_3 + 3n_4 + \ldots + (m \quad 1)n_m = r$

(iv) The number of dissimilar or distinct term in the multinomial expansion of

$$(a_1 + a_2 + \ldots + a_m)^n \text{ is } {}^{n+m\ \ 1}C_{m\ \ 1}.$$

Observe the example to find coefficient of x^n of multinomial expansion.

Find the coefficient of x^7 in expansion of $(1 + 3x \quad 2x^3)^{10}$.

Sol. Coefficient of x^7 in the expansion of $(1 + 3x \quad 2x^3)^{10}$

$$= \frac{10!}{n_1!\ n_2!\ n_3!}(1)^{n_1}(3x)^{n_2}(\ 2x^3)^{n_3}$$

where, $\quad n_1 + n_2 + n_3 = 10$ and $n_2 + 3n_3 = 7$

	$n_2 + 3n_3 = 7$		
n_1	n_2	n_3	$n_1 + n_2 + n_3$
3	7	0	10
5	4	1	10
7	1	2	10

Coefficient of x^7

$$= \frac{10!}{3!7!0!}(1)^3(3)^7(\ 2)^0 + \frac{10!}{5!4!1!}(1)^5(3)^4(\ 2)^1 + \frac{10!}{7!1!2!}(1)^7(3)^1(\ 2)^2$$

$$= 262440 \quad 204120 + 4320 = 62640$$

Binomial Theorem for any Index

Let n be a rational number and x be a real number, such that $|x| < 1$, then

$$(1 + x)^n = 1 + nx + \frac{n(n\ \ 1)}{2!}x^2 + \frac{n(n\ \ 1)(n\ \ 2)}{3!}x^3 + \ldots + \frac{n(n\ \ 1)\ (n\ \ 2)\ldots(n\ \ r+1)}{r!}x^r + \ldots$$

Note

(i) The condition $|x| < 1$ is unnecessary, if n is a whole number while the same condition is essential, if n is a rational number other than a whole number.

(ii) There are infinite number of terms in the expansion of $(1 + x)^n$, when n is a negative integer or a fraction.

(iii) If n is a positive integer, then the above expansion contains $(n + 1)$ terms and coincide with

$$(1 + x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + \ldots + {}^nC_n x^n$$

Expansion of $(x + a)^n$

To expand $(x + a)^n$, where n is the either a negative integer or a fraction, *we have following two cases*

Case I When $x > a$

Then, $\frac{a}{x} < 1$, in this case we have

$$(x + a)^n = \left[x\left(1 + \frac{a}{x}\right)\right]^n$$

$$= x^n\left(1 + \frac{a}{x}\right)^n = x^n\left[1 + n \times \frac{a}{x} + \frac{n(n\ \ 1)}{2!} \times \left(\frac{a}{x}\right)^2 + \ldots\right]$$

Case II When $x < a$

Then, $\frac{x}{a} < 1$, in this case we have

$$(x + a)^n = \left[a\left(1 + \frac{x}{a}\right)\right]^n$$

$$= a^n\left(1 + \frac{x}{a}\right)^n$$

$$= a^n\left[1 + n \times \frac{x}{a} + \frac{n(n\ \ 1)}{2!}\left(\frac{x}{a}\right)^2 + \frac{n(n\ \ 1)(n\ \ 2)}{3!}\left(\frac{x}{a}\right)^3 + \ldots\right]$$

General Term

The general term in the expansion of $(1 + x)^n$ is given by

$$T_{r+1} = \frac{n(n\ \ 1)\ (n\ \ 2)\ldots(n\ \ r+1)}{r!}x^r$$

SOME STANDARD EXPANSIONS

(i) $(1+x)^n = 1 + nx + \frac{n(n+1)}{2!}x^2 + \frac{n(n+1)(n+2)}{3!}x^3 + \ldots + (-1)^r \frac{n(n+1)(n+2)\ldots(n+r-1)}{r!}x^r + \ldots$

(ii) $(1-x)^{-n} = 1 + nx + \frac{n(n+1)}{2!}x^2 + \frac{n(n+1)(n+2)}{3!}x^3 + \ldots + \frac{n(n+1)(n+2)\ldots(n+r-1)}{r!}x^r + \ldots$

(iii) $(1-x)^{-n} = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots + \ldots + (-1)^r \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!}x^r + \ldots$

Exponential Series

Following are important results or observations in exponential series

(i) $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \ldots$, where x may be any real or complex and $\lim_{n} \left(1+\frac{1}{n}\right)^n = e$; i.e. $e = 1 + \frac{1}{1!} + \frac{1}{2!} + \ldots + \ldots$

(ii) $a^x = e^{\log_e a^x} = e^{x \log_e a}$

$\therefore a^x = 1 + \frac{x \log_e a}{1!} + \frac{(x \log_e a)^2}{2!} + \frac{(x \log_e a)^3}{3!} + \ldots + \ldots$

(iii) (a) $e + e^{-1} = 2\left(1 + \frac{1}{2!} + \frac{1}{4!} + \frac{1}{6!} + \ldots + \ldots\right)$

(b) $e - e^{-1} = 2\left(1 + \frac{1}{3!} + \frac{1}{5!} + \frac{1}{7!} + \ldots + \ldots\right)$

Logarithmic Series

Some important results based on logarithmic series are as follow

(i) $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots + \ldots$ where, $-1 < x \leq 1$

(ii) $\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \ldots$

(iii) $\ln\left(\frac{1+x}{1-x}\right) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots + \ldots\right), |x| < 1$

SPEED BOOSTER

(i) $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots + \ldots = \ln 2$

(ii) $\sum_{n=1}^{\infty} \frac{n}{n!} = e$ (iii) $\sum_{n=1}^{\infty} \frac{n^2}{n!} = 2e$

(iv) $\sum_{n=1}^{\infty} \frac{n^3}{n!} = 5e$ (v) $\sum_{n=1}^{\infty} \frac{n^4}{n!} = 15e$

Best Practice SHOTS

26. The total number of terms in the expansion of $(2x - y + 4z)^{12}$ is

(a) 90 (b) 91
(c) 13 (d) None of these

27. The coefficient of x^n in the expansion of $(1 - 9x + 20x^2)^{-1}$ is

(a) $5^n - 4^n$ (b) $5^{n+1} - 4^{n+1}$
(c) $5^{n-1} - 4^{n-1}$ (d) None of these

28. If x is very small in magnitude compared with a such that $\left(\frac{a}{a+x}\right)^{1/2} + \left(\frac{a}{a-x}\right)^{1/2} = 2 + k\frac{x^2}{a^2}$, then the value of k is

(a) $\frac{1}{4}$ (b) $\frac{1}{2}$
(c) $\frac{3}{4}$ (d) 1

29. The number of terms containing x^5 in the expansion of $(2 - x + 3x^2)^6$ is

(a) 3 (b) 4
(c) 5 (d) None of these

30. If x is very small compared with unity such that $\frac{\sqrt{1+x} + \sqrt[3]{(1-x)^2}}{\sqrt{1+x} + (1-x)} \approx a + bx$, then the values of a and b are

(a) $a = 1, b = \frac{5}{6}$ (b) $a = 1, b = -\frac{5}{6}$
(c) $a = 1, b = \frac{5}{3}$ (d) $a = 1, b = -\frac{5}{3}$

31. The coefficient of $x^3y^4z^5$ in the expansion of $(xy + yz + zx)^6$ is

(a) 120 (b) 20 (c) 30 (d) 60

32. The general term in the expansion of $(1 - 2x)^{3/4}$ is

(a) $\frac{3}{2^r r!}x^2$ (b) $\frac{3^r}{2^r r!}x^r$ (c) $\frac{3^r}{2^r 2r!}x^r$ (d) None of these

33. In the expansion of $(2x + y + 3z)^{10}$, the coefficient of $x^2y^5z^3$ is

(a) $\frac{10!}{2!\,3!\,5!}$ (b) $\frac{10!}{2!\,3!\,5!}2^2\,3^3$
(c) $\frac{10!}{2!\,3!\,5!}2^3\,3^2$ (d) $10!\,2^2\,3^3$

34. The coefficient of x^4 in the expansion of $\left(\frac{1}{1-x}+x\right)^2$ is

(a) n (b) $2n$ (c) $3n$ (d) $4n$

35. If x is so small that x^3 and higher powers of x may be neglected, then $\frac{(1+x)^{3/2}\left(1+\frac{x}{2}\right)^3}{(1-x)^{1/2}}$ may be approximated as [AIEEE 2005]

(a) $\frac{x}{2}-\frac{3}{8}x^2$ (b) $\frac{3}{8}x^2$ (c) $3x+\frac{3}{8}x^2$ (d) $1-\frac{3}{8}x^2$

36. The coefficient of x^6 in the expansion of $(1+x+x^2)^{-3}$ is

(a) 6 (b) 5 (c) 4 (d) 3

37. The sum of the series
${}^4C_0+{}^5C_1x+{}^6C_2x^2+{}^7C_3x^3+\ldots+$ is

(a) $(1-x)^{-4}$ (b) $\frac{1}{(1-x)^5}+(1+x)^5$

(c) $(1+x)^4$ (d) None of these

38. If $\frac{x^2+x}{x-1}=a_1x+a_2x^2+\ldots+$, $|x|<1$, then

(a) $a_1+a_2=4$ (b) $a_1-a_2=3$
(c) $a_p=a_q$ for $p,q>1$ (d) None of these

39. The coefficient of x^r in the expansion of $(1-4x)^{-1/2}$ is

(a) $\frac{(2r)!}{r!2^r}$ (b) ${}^{2r}C_r$

(c) $\frac{1\times3\times5\ldots(2r-1)}{2^r\times r!}$ (d) None of these

40. The coefficient of x^6 in the expansion of $(1+x^2-x^3)^8$ is

(a) 80 (b) 84 (c) 88 (d) 92

41. The greatest coefficient in the expansion of $(x+y+z+t)^{18}$ is

(a) $\frac{18!}{4!5!}$ (b) $\frac{18!}{(4!)^2\,5!}$ (c) $\frac{18!}{(5!)^2\,4!}$ (d) $\frac{18!}{(5!)^2(4!)^2}$

42. The number of rational term in the expansion of $(1+\sqrt{2}+\sqrt[3]{3})^6$ is

(a) 6 (b) 7 (c) 5 (d) 8

43. Assuming x to be so small that x^2 and higher powers of x can be neglected, then

$\frac{\left(1+\frac{3}{4}x\right)^{-4}(16-3x)^{1/2}}{(8+x)^{2/3}}$ may equal to

(a) $1+\frac{305}{96}x$ (b) $1-\frac{305}{96}x$ (c) $1+\frac{96}{305}x$ (d) None

44. The total number of dissimilar terms in the expansion of $(x_1+x_2+\ldots+x_n)^3$, is

(a) n^3 (b) $\frac{n^3+3n^2}{4}$

(c) $\frac{n(n+1)(n+2)}{6}$ (d) $\frac{n^2(n+1)^2}{4}$

45. The coefficient of x^3 in the expansion of $(1-x+x^2)^5$ is

(a) 10 (b) 20 (c) 50 (d) 30

46. In the expansion of $2\log_e x-\log(x+1)-\log_e(x-1)$, the coefficient of x^{-4} is

(a) $\frac{1}{2}$ (b) 1 (c) 1 (d) None

47. The expansion of $\log(1+3x+2x^2)$ is

(a) $3x-\frac{5}{4}x^2+\frac{9}{3}x^3-\frac{17}{4}x^4+\ldots+$

(b) $4x-\frac{5}{4}x^2+\frac{9}{3}x^3-\frac{17}{4}x^4+\ldots+$

(c) $3x-\frac{5}{2}x^2+\frac{9}{3}x^3-\frac{17}{4}x^4+\ldots+$

(d) $3x-\frac{5}{4}x^2-\frac{9}{3}x^3-\frac{17}{4}x^4+\ldots+$

48. The coefficient of x^k in the expansion of $\frac{1-2x-x^2}{e^x}$ is

(a) $\frac{1-k-k^2}{k!}$ (b) $\frac{k^2+1}{k!}$ (c) $\frac{1-k}{k!}$ (d) $\frac{1}{k!}$

49. The coefficient of x^n in the expansion of $\frac{(a-bx)}{e^x}$ is

(a) $\frac{(-1)^n}{n!}(a+bn)$ (b) $\frac{(-1)^n}{n!}(b+an)$

(c) $\frac{(-1)^{n+1}}{n!}(a+bn)$ (d) None of these

50. The coefficient of x^3 in the expansion of 3^x is

(a) $\frac{3^3}{6}$ (b) $\frac{(\log 3)^3}{3}$ (c) $\frac{\log(3)^3}{6}$ (d) $\frac{(\log 3)^3}{6}$

Notice Board

WBJEE will be on May 17, 2016

West Bengal Joint Entrance Examinations Board (WBJEE Board) has released information brochure of WBJEEM 2016 on December 24, 2016. According to the WEBJEEM 2016 brochure, the Application form for WBJEEM 2016 will be available from January 5, 2016 onwards. WBJEEM 2016 will be conducted by the board on May 17, 2016 for admission to undergraduate level professional courses in Engineering, Technology, Architecture and Pharmacy in government colleges and self-financed institutes in West Bengal.The information brochure contains important details related to the entrance exam like Question Pattern, Ranking and tie breaking, Eligibility criteria for candidates, Availability of seats, Application Procedure, Important Dates, Admit Card¸ Counselling and Admission, Syllabus etc.

MATHEMATICAL INDUCTION

The word induction means the generalisation from particular cases or facts. In algebra or in other disciplines of Mathematics there are certain results or statements that are formulated in terms of n, where n is a positive integer. To prove such statements the well suited principle which is based on specific technique is the principle of Mathematical induction.

Statements

A sentence or description which can be judged to be true or false is called statements.

Mathematical Statements

Statements involving Mathematical relations are know as the Mathematical statements.

Peano's Theorem

The statement or proposition $P(n)$ is assumed to be true for all values of n N

(a) $P(1)$ is true (b) $P(m)$ is true $\Rightarrow$ $P(m+1)$ is true.

Principle of Mathematical Induction

Let $P(n)$ be a statement involving all natural numbers n such that

(i) The statement is true for $n = 1$, i.e. $P(1)$ is true

(ii) If the statement is true for $n = k$, then the statement is also true for $n = k + 1$, i.e. truth of $P(k)$ = truth of $P(k + 1)$, where k is some positive integer.

Then, $P(n)$ is true for all natural numbers n.

In order to prove the validity of the statements we use following steps

Step I Verification step Actual verification of the proposition for the starting value '1'.

Step II Induction step Assuming the proposition to be true for 'k', k 1 and then prove that it is true for the value $(k + 1)$ which is the next higher integer.

Step III Generalisation step Combine the above two steps.

Best Practice SHOTS

51. For all n N, $2 \times 4^{2n+1} + 3^{3n+1}$ is divisible by

(a) 7 (b) 5 (c) 11 (d) 209

52. $\frac{3}{4} + \frac{15}{16} + \frac{63}{64} + \ldots$ to n terms is equal to

(a) n $\frac{4^n}{3}$ $\frac{1}{3}$ (b) $n + \frac{4^n}{3}$ $\frac{1}{3}$ (c) $n + \frac{4^n}{3}$ $\frac{1}{3}$ (d) n $\frac{4^n}{3} + \frac{1}{3}$

53. For all odd positive interger n, the number $n(n^2$ 1) is divisible by

(a) 36 (b) 9 (c) 16 (d) 24

54. $\frac{1}{3 \times 5} + \frac{1}{5 \times 7} + \frac{1}{7 \times 9} + \ldots$ to n term is equal to

(a) $\frac{n}{3(2n+3)}$ (b) $\frac{n}{(2n+3)}$

(c) $\frac{1}{(n+2)(n+1)}$ (d) None of these

55. The product of three consecutive natural numbers is divisible by

(a) 3 (b) 8 (c) 6 (d) 11

56. Let $S(k) = 1 + 3 + 5 + \ldots + (2k \quad 1) = 3 + k^2$. Then, which of the following is true? **[AIEEE 2004]**

(a) Principle of mathematical induction can be used to prove the formula

(b) $S(K) \Rightarrow S(K + 1)$

(c) $S(K) \not\Rightarrow S(K + 1)$ (d) $S(1)$ is correct

57. The sum of the series $1 + 2 + 3 + 4 + \ldots + n$ is less than

(a) $\frac{1}{8}(2n+1)^2$ (b) $\frac{1}{12}(2n+1)^2$

(c) $\frac{1}{8}(2n+1)$ (d) None of these

58. If $u_{n+1} = 3u_n \quad 2u_{n-1}$ and $u_0 = 2, u_1 = 3$, then u_n is equal to

(a) 1 2^n (b) $2^n + 1$

(c) 2^n 1 (d) $2^n + 2$

59. The remainder when 5^{99} is divisible by 13 is

(a) 6 (b) 8 (c) 9 (d) 10

60. For all n N $\int_0^{\pi} \frac{\sin(2nx)}{\sin x}\,dx$ is equal to

(a) 0 (b) (c) $\frac{}{2}$ (d)

61. If $49^n + 16^n + k$ is divisible by 64 for n N, then the least negative integral value of k is

(a) 1 (b) 2

(c) 3 (d) 4

62. If $A = \begin{matrix} \cos & \sin \\ \sin & \cos \end{matrix}$, then for n N, A^n is equal to

(a) $\begin{matrix} \cos^n & \sin^n \\ \sin^n & \cos^n \end{matrix}$ (b) $\begin{matrix} \cos n & \sin n \\ \sin n & \cos n \end{matrix}$

(c) $\begin{matrix} n\cos & n\sin \\ n\sin & n\cos \end{matrix}$ (d) None of these

Answers

1. (b)	**2.** (b)	**3.** (c)	**4.** (c)	**5.** (d)	**6.** (d)	**7.** (b)	**8.** (b)	**9.** (d)	**10.** (c)
11. (a)	**12.** (c)	**13.** (b)	**14.** (b)	**15.** (a)	**16.** (c)	**17.** (c)	**18.** (a)	**19.** (b)	**20.** (b)
21. (c)	**22.** (d)	**23.** (c)	**24.** (b)	**25.** (c)	**26.** (b)	**27.** (b)	**28.** (c)	**29.** (a)	**30.** (b)
31. (d)	**32.** (d)	**33.** (b)	**34.** (d)	**35.** (b)	**36.** (d)	**37.** (b)	**38.** (c)	**39.** (b)	**40.** (b)
41. (d)	**42.** (b)	**43.** (b)	**44.** (c)	**45.** (d)	**46.** (a)	**47.** (c)	**48.** (a)	**49.** (a)	**50.** (d)
51. (c)	**52.** (b)	**53.** (d)	**54.** (a)	**55.** (c)	**56.** (b)	**57.** (a)	**58.** (b)	**59.** (b)	**60.** (a)
61. (a)	**62.** (b)								

Formula at a Glance

1. If x and a are real numbers, n N, then

$(x + a)^n = {}^nC_0x^na^0 + {}^nC_1x^{n\ 1}a^1 + {}^nC_2x^{n\ 2}a^2 + \ldots$

$+ {}^nC_rx^{n\ r}a^r + \ldots + {}^nC_{n\ 1}x^1a^{n\ 1} + {}^nC_nx^0a^n$

General term $T_{r+1} = {}^nC_rx^{n\ r}a^r$

2. $(x\ \ a)^n = {}^nC_0x^na^0\ \ {}^nC_1x^{n\ 1}a^1 + {}^nC_2x^{n\ 2}a^2$

$\ldots + (\ 1)^n\ {}^nC_nx^0a^n$

General term, $T_{r+1} = (\ 1)^r\ {}^nC_rx^{n\ r}a^r$

3. In the binomial expansion of $(x + a)^n$, the rth term from the end is $(n\ \ r + 2)$th term from the beginning.

4. Middle term in the binomial expansion $(x + a)^n$ is/are

(i) $\left(\frac{n}{2} + 1\right)$th term, if n is even, i.e. $T_{\frac{n}{2}+1} = {}^nC_{n/2}x^{n/2}a^{n/2}$

(ii) $\left(\frac{n+1}{2}\right)$th term and $\left(\frac{n+3}{2}\right)$th term, if n is odd,

i.e. $T_{\frac{n+1}{2}} = {}^nC_{\frac{n\ 1}{2}}x^{\frac{n+1}{2}}a^{\frac{n\ 1}{2}}$

and $T_{\frac{n+3}{2}} = {}^nC_{\frac{n+1}{2}}x^{\frac{n\ 1}{2}}a^{\frac{n+1}{2}}$

5. Greatest term in the expansion of $(x + a)^n$ is

$\frac{T_{r+1}}{T_r}\ \ 1$, i.e. $\frac{n\ \ r+1}{r}\left|\frac{a}{x}\right|\ \ 1$

Binomial coefficient of middle term is the greatest binomial coefficient.

6. $(x + a)^n + (x\ \ a)^n = 2[{}^nC_0\ x^na^0 + {}^nC_2\ x^{n\ 2}a^2 + \ldots]$

$(x + a)^n\ \ (x\ \ a)^n = 2[{}^nC_1\ x^{n\ 1}a^1 + {}^nC_3\ x^{n\ 3}a^3 + \ldots]$

7. Let n be a rational number and x be a real number such that $|x| < 1$, then

$$(1 + x)^n = 1 + nx + \frac{n(n\ \ 1)}{2!}x^2 + \frac{n(n\ \ 1)(n\ \ 2)}{3!}x^3 + \ldots$$

$$+ \frac{n(n\ \ 1)(n\ \ 2)\ldots\{n\ \ (r\ \ 1)\}}{r!}x^r + \ldots$$

General term in the expansion of $(1 + x)^n$,

$$T_{r+1} = \frac{n(n\ \ 1)(n\ \ 2)\ldots\{n\ \ (r\ \ 1)\}}{r!}x^r$$

8. ${}^nC_r = \frac{n}{r}{}^{n\ 1}C_{r\ 1} = \frac{n}{r}\times\frac{n\ \ 1}{r\ \ 1}{}^{n\ 2}C_{r\ 2}$ and so on.

9. (i) ${}^nC_r + {}^nC_{r\ 1} = {}^{n+1}C_r$

(ii) $C_0 + C_1 + C_2 + \ldots + C_n = 2^n = \sum_{r=0}^{n} {}^nC_r$

(iii) $C_0 + C_2 + C_4 + \ldots = C_1 + C_3 + C_5 + \ldots = 2^{n\ 1}$

(iv) $C_0\ \ C_1 + C_2\ \ C_3 + \ldots + (\ 1)\ {}^nC_n = 0 = \sum_{r=0}^{n}(\ 1)^r\ {}^nC_r$

(v) ${}^nC_1\ \ 2\ {}^nC_2 + 3\ {}^nC_3\ \ \ldots + n(\ 1)^{n\ 1}\ {}^nC_n = 0$

(vi) ${}^nC_1 + 2{}^nC_2 + 3{}^nC_3 + \ldots + n{}^nC_n = n\,2^{n\ 1}$

(vii) $C_0C_r + C_1C_{r+1} + \ldots + C_{n\ r}C_n = \frac{(2n!)}{(n\ \ r)!\,(n + r)!}$

(viii) $C_0^2 + C_1^2 + C_2^2 + \ldots + C_n^2 = \frac{(2n)!}{(n!)^2}$

(ix) $C_0^2\ \ C_1^2 + C_2^2\ \ \ldots = \begin{cases} 0, \text{ if } n \text{ is odd} \\ (\ 1)^{n/2}\ {}^nC_{n/2}, \text{if } n \text{ is even} \end{cases}$

(x) ${}^nC_n + {}^{n+1}C_n + {}^{n+2}C_n + \ldots + {}^{2n\ 1}C_n = {}^{2n}C_{n+1}$

10. **Multinomial Theorem**

If n is positive integer and $x_1, x_2, \ldots\ \ R$, then

$$(x_1 + x_2 + \ldots + x_k)^n = \sum_{r_1 + r_2 + \ldots + r_k = n} \frac{n!\,x_1^{r_1}\,x_2^{r_2}\ldots x_k^{r_k}}{r_1!\,r_2!\ldots r_k!}$$

11. (i) Number of terms in $(x + y + z)^n$ is ${}^{n+2}C_2$ or $\frac{(n+1)(n+2)}{2}$

(ii) Coefficient of $x^{n_1}.\ y^{n_2} \times z^{n_3}$ in the expansion of $(x + y + z)^n$ is $\frac{n!}{n_1!\,n_2!\,n_3!}$, where $n = n_1 + n_2 + n_3$.

12. (i) $(1 + x)^{\ 1} = 1\ \ x + x^2\ \ x^3 + \ldots + (\ x)^r + \ldots$

(ii) $(1\ \ x)^{\ 1} = 1 + x + x^2 + x^3 + \ldots + x^r + \ldots$

(iii) $(1 + x)^{\ 2} = 1\ \ 2x + 3x^2\ \ 4x^3 + \ldots + (r + 1)(\ x)^r + \ldots$

(iv) $(1\ \ x)^{\ 2} = 1 + 2x + 3x^2 + 4x^3 + \ldots + (r + 1)x^r + \ldots$

(v) $(1 + x)^{\ 3} = 1\ \ 3x + 6x^2\ \ \ldots + \frac{(r+1)(r+2)}{2!}(\ x)^r + \ldots$

(vi) $(1\ \ x)^{\ 3} = 1 + 3x + 6x^2 + \ldots + \frac{(r+1)(r+2)}{2!}(x)^r + \ldots$

Permutation and Combination

Permutation and combination are important concept for JEE Main and Advanced, good understanding of this chapter will also help you in probability. Generally questions based on permutation are asked in probability. Students are advised to read the questions properly. Since, a little bit change in the interpretation of question can lead you to wrong answer. Atleast one question is asked every year in JEE.

PERMUTATION

Factorial Notation

Let n be a positive integer, then continued product of first n natural numbers is called factorial n and is denoted by $n!$ or $\lfloor n$.

$\therefore\ n! = n\,(n-1)\,(n-2)\ldots 3\cdot 2\cdot 1$

Note (i) $0! = 1$ (ii) $n! = n\,(n-1)!$

Fundamental Principle of Counting

There are two fundamental principles of counting.

(i) Multiplication Rule

If a task can be done by m different ways and in continuation and other task can be done by n different ways. Then, total number of different ways of simultaneous completion of the task is $m \times n$

or

If a job has n parts and job will be completed only when each part get completed. Let first part can be completed in p_1 ways, the second can be completed in p_2 ways and so on and nth part can be completed in p_n ways. Then, total number of ways of doing the job is

$$p_1 \times p_2 \times p_3 \times \ldots \times p_n$$

(ii) Addition Rule

If a task can be completed in m different ways and another task can be completed in n different ways. Then, exactly one of the task can be completed by $(m+n)$ possible ways.

REMEMBER

- All the numbers whose last digit is an even number 0, 2, 4, 6 or 8, are divisible by 2.
- If sum of the digits is divisible by 3, then number is divisible by 3 and if sum is divisible by 9, then number is divisible by 9.
- If last two digits are divisible by 4, then the number is divisible by 4. If last two digits are divisible by 25, then number is divisible by 25.
- If a number ends with 0 and 5, then number is always divisible by 5.
- If a number is divisible by 2 and 3, then the number is divisible by 6.
- If last three digits of a number are divisible by 8, then the number is divisible by 8.

Permutation

Permutation is arrangement of objects. Basically, it involves two steps, i.e. selection and then arrangement. When we arrange the objects, we are concern about their order. e.g. Arrangement of the letters a,b,c taking two at a time are ab, bc, ca, ba, cb, ac.

Meaning of ${}^nP_r$

Number of permutations of n distinct objects taking r at a time is denoted by ${}^nP_r$

$${}^nP_r = \frac{n!}{n-r!},\ \forall\ 0 \le r \le n$$

$$= n\,(n-1)\,(n-2)\ldots(n-r+1),\ \forall\ n \in N \text{ and } r \in W$$

Properties of ${}^nP_r$

(i) ${}^nP_0 = 1,\ {}^nP_1 = n$ and ${}^nP_{n-1} = n!$

(ii) ${}^nP_r = n \cdot {}^{n-1}P_{r-1} = r \cdot {}^{n-1}P_{r-1} + {}^{n-1}P_r$

(iii) ${}^{n-1}P_r = (n-r)\,{}^{n-1}P_{r-1}$

Different Cases of Permutation

(i) The number of permutation of n objects taken 'r' at a time when repetition is not allowed is ${}^nP_r = \frac{n!}{n-r!}$

(ii) The number of permutations of n distinct object taken all at a time is ${}^nP_n = n!$

(iii) The number of permutations of n different objects taken r at a time when repetition of objects are allowed, is equal to n^r.

Permutation of Objects not all Distinct

Number of arrangements of n objects when p particular objects are identical of one kind, q particular objects are identical of other kind and r particular objects are identical of third kind, such that $p+q+r=n$, then

$$\text{Number of permutations} = \frac{n!}{p!\,q!\,r!}$$

In general, the number of permutations of n things, of which p_1 are alike of one kind, p_2 are alike of second kind, p_3 are alike of third kind $\ldots p_r$ are alike of rth kind such that

$$p_1 + p_2 + p_3 + \ldots p_r = n \text{ is } \frac{n!}{p_1!\,p_2!\,p_3!\ldots p_r!}$$

SPEED BOOSTER

(i) If we are given n different digits $(n_1, n_2, \ldots, n_n)$, then sum of the digits in the unit place of all numbers formed without repetition is $(n-1)!(n_1 + n_2 + \ldots + n_n)$.

(ii) Sum of the total numbers formed by n digits $(n_1, n_2, n_3, \ldots, n_n)$ is $(n-1)!(n_1 + n_2 + n_3 + \ldots + n_n)(111\ldots n \text{ times})$.

(iii) The number of ways in which n (one type of different things) and m (another types of different things) can be arranged in a row alternatively is $2mn!$, here $(m = n)$.

Permutations under Restrictions

(i) If we place n objects in a row, then number of positions (gaps) are $(n+1)$ when question is asking to made an alternate arrangement, then place one kind of object and then select the gap to place another kind of objects.

(ii) Number of permutations of n different things taken r at a time, when particular things are to be always included in each arrangement, is $r \cdot {}^{n-1}P_{r-1}$.

(iii) Number of permutations of n different things taken r at a time, when s particular things are to be always included in each arrangement, is $s\,[r-(s-1)]^{ns}P_{r-s}$.

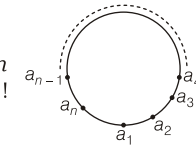
(iv) Number of permutations of n different things taken r at a time, when particular things are never taken in each arrangement is ${}^{n-1}P_r$.

(v) Number of permutations of n different things taken all at a time, when m specified things always come together, is $m! \times (n-m+1)!$.

(vi) Number of permutations of n different things taken all at a time, when m specified things never come together, is $n! - m!(n-m+1)!$.

Circular Permutation

The number of circular permutations of n different things taken all at a time is $(n-1)!$



Note (i) If anti-clockwise and clockwise order of arrangement are not distinct (e.g. arrangement of beads in a necklace, arrangement of flowers in garland etc.), then number of circular permutations, n different things taken all at a time is $\frac{1}{2}(n-1)!$.

(ii) Number of circular permutations of n different things taken r at a time, when clockwise and anti-clockwise orders are taken as different, is $\frac{{}^nP_r}{r}$.

(iii) Number of circular permutations of n different things taken r at a time, when clockwise and anti-clockwise orders are not different is $\frac{{}^nP_r}{2r}$.

SPEED BOOSTER

The number of ways in which m things of one type and n things of another type can be arranged in the form of garland, so that all the second type of things come together is $\frac{m!n!}{2}$ and no two things of second type come together is $\frac{(m-1)\,{}^mP_n}{2}$.

Exponent of Prime p in $n!$

Exponent of a prime p in $n!$ is denoted by $E_p(n!)$ and is given by

$$E_p(n!) = \left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + \left[\frac{n}{p^3}\right] + \ldots$$

Best Practice SHOTS

1. The value of $1\cdot1! + 2\cdot2! + 3\cdot3! + \ldots + n\cdot n!$ is

(a) $(n+1)!$ (b) $(n+1)! + 1$
(c) $(n+1)! - 1$ (d) None of these

2. If ${}^{n-1}C_r = (k^2-3)\,{}^nC_{r+1}$, then k belongs to the interval

(a) $[-\sqrt{3}, \sqrt{3}]$ (b) $(-\infty, -2)$
(c) $(2, \infty]$ (d) $(\sqrt{3}, 2]$

3. The value of the expression ${}^{47}C_4 + \sum_{j=1}^{5} {}^{52-j}C_3$ is

(a) ${}^{52}C_4$ (b) ${}^{52}C_3$
(c) ${}^{51}C_4$ (d) ${}^{53}C_4$

4. The value of ${}^{50}C_4 + \sum_{r=1}^{6} {}^{56-r}C_3$ is [AIEEE 2005]

(a) ${}^{56}C_4$ (b) ${}^{56}C_3$
(c) ${}^{55}C_3$ (d) ${}^{55}C_4$

5. Five persons entered the lift cabin on the ground floor of an 8 floor house. Suppose each of them can leave the cabin independently at any floor beginning with the first. The total number of ways in which each of the five persons can leave the cabin at any one of the 7 floors, is

(a) 5^7 (b) 2520 (c) 7^5 (d) 35

6. A rectangle with sides of lengths $(2n-1)$ and $(2m-1)$ units is divided into squares of unit length. The number of rectangles which can be formed with sides of odd length, is

(a) m^2n^2 (b) $mn\,(m+1)\,(n+1)$
(c) 4^{m+n-1} (d) None of these

7. How many 5 letter words, with or without meaning, can be formed out of the letters of the word 'EQUATION', if repetition of letters is not allowed?

(a) 126 (b) 5^9 (c) 9^5 (d) 15120

8. Ten different letters of an alphabet are given. Words with five letters are formed from these given letters. The number of words which have atleast one letter repeated, is

(a) 10^5 (b) 30240
(c) 69760 (d) None of these

9. The sum of all the numbers greater than 10000 formed by using digits 0, 2, 4, 6, 8, no digit being repeated in any number, is

(a) $\frac{20}{9}(10^5 - 10^4)$

(b) $\frac{40}{3}\{4(10^5 - 1) - (10^4 - 1)\}$

(c) $\frac{20}{9}\{4(10^5 - 1) - (10^4 - 1)\}$

(d) $20 \times 4!\left(\frac{10^5 - 1}{10 - 1}\right)$

10. The total number of natural number lying between 20000 and 60000 the sum of whose digits is even, is

(a) 40000 (b) 20000
(c) 5000 (d) 25000

11. Five boys and five girls make a line with the boys and girls alternating. The number of ways of making the line, is

(a) $(5!)^2$ (b) 10!
(c) 5! + 5! (d) $2(5!)^2$

12. How many different 9-digit numbers can be formed from the number 223355888 by rearranging its digit, so that the odd digits occupy even positions?

(a) 16 (b) 36
(c) 60 (d) 180

13. Eleven animals of a circus have to be placed in eleven cages (one in each cage). If 4 of the cages are too small for 6 of the animals, then the number of ways of caging all the animals is

(a) 60480 (b) 5980
(c) 604800 (d) None of these

14. There are six teachers. Out of them two are primary teachers, two are middle teachers and two secondary teachers. They are to stand in a row, so as the primary teachers, middle teachers and secondary teachers are always in a set. Then, the number of ways in which they can do so is

(a) 48 (b) 50
(c) 55 (d) 65

15. Eight chair are numbered 1 to 8. Two women and three men wish to occupy one chair each. First the women choose the chairs from amongst the chair marked 1 to 4 and then the men select the chairs from amongst the remaining. The number of possible arrangements is

(a) ${}^6P_2 \times {}^4P_2$ (b) ${}^4P_2 \times {}^6P_3$
(c) ${}^4P_2 + {}^6P_3$ (d) None of these

16. The letters of the word SACHIN are arranged in all possible ways and these words are written out as in dictionary, then the word SACHIN appears at serial number

(a) 602 (b) 603
(c) 600 (d) 601

17. Eight different letters of an alphabet are given. Words of four letters from these are formed. The number of such words with atleast one letter repeated is

(a) $\binom{8}{4} - {}^8P_4$ (b) $8^4 + \binom{8}{4}$

(c) $8^4 - {}^8P_4$ (d) $8^4 - \binom{8}{4}$

18. The number of ways in which 5 ladies and 7 gentlemen can be seated in a round table, so that no two ladies sit together, is

(a) $\frac{7}{2}(720)^2$ (b) $7(360)^2$
(c) $7(720)^2$ (d) 720

19. 20 persons are invited for a party. In how many different ways can they and the host be seated at circular table, if the two particular persons are to be seated on either side of the host?

(a) 20! (b) $2! \times 18!$
(c) 18! (d) None of these

20. The number of ways in which 6 men and 5 women can dine at a round table, if no two women are to sit together, is

(a) $6! \times 5!$ (b) 30
(c) $5! \times 4!$ (d) $7! \times 5!$

21. Six persons A, B, C, D, E, F are to be seated at a circular table. If A should have either B or C on his right and B must always have either C or D on his right, then the number of ways this can be done is

(a) 18 (b) 20
(c) 22 (d) 24

22. The number of ways in which 8 different flowers can be strung to form a garland, so that four particular flowers are never separated, is

(a) 240 (b) 280
(c) 288 (d) 390

23. The number of zeroes at the end of 70! is

(a) 16 (b) 5
(c) 7 (d) 70

24. The exponent of 7 in 100! is

(a) 14 (b) 15
(c) 16 (d) None of these

COMBINATION

Combination

Each of the different selections made by taking some or all of a number of distinct objects or items, irrespective of their arrangements or order in which they are placed, is called a combination. Consider the first three letters of the English alphabet. All possible combination (selection) of these three letters taken two at a time are *ab*, *bc*, *ca*. Here, selection *ab* is same as *ba*, similarly, selections *bc* and *ca* are same as *cb* and *ac*, respectively.

The number of combinations of n different things taken r at a time, is denoted as ${}^nC_r$, where ${}^nC_r = \dfrac{n!}{(n-r)!\,r!}$

Also, ${}^nC_r = \dfrac{{}^nP_r}{r!} \Rightarrow {}^nP_r = {}^nC_r \cdot r!$

Properties of Combination

(i) If ${}^nC_r = {}^nC_p$, then either $r = p$ or $r + p = n$

(ii) ${}^nC_r = \dfrac{{}^nP_r}{r!}$

(iii) ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$

(iv) ${}^nC_0 + {}^nC_1 + {}^nC_2 + \ldots + {}^nC_n = 2^n$

(v) ${}^nC_0 + {}^nC_2 + {}^nC_4 + \ldots = {}^nC_1 + {}^nC_3 + \ldots = 2^{n-1}$

(vi) ${}^nC_r = \dfrac{n}{r}\,{}^{n-1}C_{r-1} = \dfrac{n}{r}\cdot\dfrac{n-1}{r-1}\cdot{}^{n-2}C_{r-2}$

(vii) ${}^{2n+1}C_0 + {}^{2n+1}C_1 + {}^{2n+1}C_2 + \ldots + {}^{2n+1}C_n = 2^{2n}$

(viii) ${}^nC_n + {}^{n+1}C_n + {}^{n+2}C_n + \ldots + {}^{2n-1}C_n = {}^{2n}C_{n+1}$

Restricted Combination

Number of combinations of n different things taken r at a time

(i) When p particular things are always included $= {}^{n-p}C_{r-p}$

(ii) When p particular things are never included $= {}^{n-p}C_r$

(iii) When p particular things are not together in any selection

$= {}^nC_r - {}^{n-p}C_{r-p}$

SPEED BOOSTER

(i) Number of selections of r consecutive things out of n things along a circle $= \begin{cases} n, & \text{when } r < n \\ 1, & \text{when } r = n \end{cases}$

(ii) Number of selections of r consecutive things out of n things in a row $= (n - r + 1)$

Selection of One or More Objects

When we select zero or more objects from given number of objects. Then, following cases are possible

Selection from Different Objects

(i) Number of selections of zero or more things out of n different things, is ${}^nC_0 + {}^nC_1 + {}^nC_2 + \ldots + {}^nC_n = 2^n$

(ii) Number of selections of atleast one or more things out of n different things is ${}^nC_1 + {}^nC_2 + {}^nC_3 + \ldots + {}^nC_n = 2^n - 1$

Selection from Identical Objects

(i) The number of selections of r objects out of n identical objects is 1.

(ii) Total number of selections of zero or more objects from n identical objects is $(n + 1)$.

(iii) The total number of selections of atleast one out of $a_1 + a_2 + \ldots + a_n$ objects, where a_1 objects are alike of one kind and a_2 objects are alike of second kind and so on ..., a_n objects are alike of nth kind, is $(a_1 + 1)(a_2 + 1)\ldots(a_n + 1) - 1$

Selections when both Identical and Distinct Objects are Present

(i) The number of possible combinations of n objects of which p are identical taken r at time is

$$\begin{cases} {}^{n-p}C_r + {}^{n-p}C_{r-1} + {}^{n-p}C_{r-2} + \ldots + {}^{n-p}C_0, & \text{if } r \le p \\ {}^{n-p}C_r + {}^{n-p}C_{r-1} + {}^{n-p}C_{r-2} + \ldots + {}^{n-p}C_{r-p}, & \text{if } r \ge p \end{cases}$$

(ii) The number of selections taking atleast one out of $a_1 + a_2 + a_3 + \ldots + a_n + k$, objects, where a_1 are alike of one kind, a_2 are alike of second kind ..., a_n are alike of nth kind and k are distinct objects, is $(a_1 + 1)(a_2 + 1)\ldots(a_n + 1)\cdot 2^k - 1$

Division into Groups and Distribution

(i) Number of ways of dividing $(m + n + p)$ different objects in three groups containing m, n and p objects respectively $(m \ne n \ne p)$ is

$$\begin{cases} \dfrac{(m+n+p)!}{m!n!p!}, & \text{if order is not considered} \\ \dfrac{(m+n+p)!}{m!n!p!}3!, & \text{if order of groups is considered} \end{cases}$$

(ii) Number of ways of dividing $(m + n + p)$ different objects in three groups containing m, n and p objects respectively $(m = n = p)$ is

$$\begin{cases} \dfrac{(m+n+p)!}{m!n!p!3!} = \dfrac{(3m)!}{(m!)^3 3!} & \text{order of group is not considered} \\ \dfrac{(m+n+p)!3!}{m!n!p!3!} = \dfrac{(3m)!}{(m!)^3} & \text{order of group is considered} \end{cases}$$

(iii) Number of ways of dividing n identical objects into r groups, if blank groups are allowed, is ${}^{n+r-1}C_{r-1}$.

(iv) Number of ways of dividing n identical things into r groups, if blank groups are not allowed, is ${}^{n-1}C_{r-1}$.

(v) Number of ways of dividing n identical things into r groups such that no group contains less than m things and more than $k\,(m<k)$ things is coefficient of x^n in the expansion of $(x^m + x^{m+1} + \ldots + x^k)^r$.

(vi) The number of ways of selecting r things out of n things of which p are alike of one kind and q are alike of another kind, s are alike of third kind and so on, is coefficient of x^r in expansion of

$$[(1 + x + x^2 + \ldots + x^p)(1 + x + x^2 + \ldots + x^q)(1 + x + x^2 + \ldots + x^s)\ldots]$$

when each thing is considered atleast once = coefficient of x^{r-3} in expansion of

$$(1 + x + x^2 + \ldots + x^{p-1})(1 + x + x^2 + \ldots + x^{q-1})(1 + x + x^2 + \ldots + x^{s-1})$$

(vii) The number of ways in which n different things can be distributed into r different groups, is

$$r^n - {}^rC_1(r-1)^n + {}^rC_2(r-2)^n - \ldots + (-1)^{r-1}\,{}^nC_{r-1}$$

(viii) The number of ways of selecting r things out of n things of which p are alike of one kind, q are alike of second kind and $(n-p-q)$ things are all different, is coefficient of

$$x^r \text{ in } (1 + x + x^2 + \ldots + x^p)(1 + x + x^2 + \ldots + x^q)(1 + x)^{n-p-q}$$

(ix) The number of ways in which r identical things can be distributed among n persons when each person can get 0 or more things

$$= \text{Coefficient of } x^r \text{ in } (1 + x + x^2 + \ldots + x^r)^n$$
$$= \text{Coefficient of } x^r \text{ in } (1 - x)^{-n}$$
$$= {}^{n+r-1}C_r \text{ or } {}^{n+r-1}C_{n-1}$$

SPEED BOOSTER

(i) The number of non-negative integral solution of the equation $x_1 + x_2 + x_3 + \ldots + x_r = n$ is ${}^{n+r-1}C_{r-1}$

(ii) The number of positive integral solution of the equation $x_1 + x_2 + \ldots + x_r = n$ is ${}^{n-1}C_{r-1}$.

(iii) The number of terms in the expansion of $(a_1 + a_2 + a_3 + \ldots + a_n)^r$ is ${}^{n+r-1}C_{r-1}$.

Number of Divisors (Prime Factors)

Let N be a number which can be expressed as

$$N = p_1^{a_1} \cdot p_2^{a_2} \ldots p_k^{a_k},$$

where, $p_i,\ i = 1, 2, \ldots, k$ are prime numbers.

$a_i,\ i = 1, 2 \ldots k$ are positive integers. Then,

(i) The total number of divisors of N including 1 and N is $(a_1 + 1)(a_2 + 1)\ldots(a_k + 1)$

(ii) The total number of divisors excluding 1 and N is $(a_1 + 1)(a_2 + 1)\ldots(a_k + 1) - 2$

(iii) The total number of divisors excluding 1 or N is $(a_1 + 1)(a_2 + 1)(a_3 + 1)\ldots(a_k + 1) - 1$

(iv) Sum of the distinct positive integral divisors of N is

$$\left(\frac{p_1^{a_1+1} - 1}{p_1 - 1}\right) \cdot \left(\frac{p_2^{a_2+1} - 1}{p_2 - 1}\right)\left(\frac{p_3^{a_3+1} - 1}{p_3 - 1}\right) \ldots \left(\frac{p_k^{a_k+1} - 1}{p_k - 1}\right)$$

(v) The number of ways in which N can be resolved as a product of two factors is

$$\begin{cases} \frac{1}{2}(a_1 + 1)(a_2 + 1)\ldots(a_k + 1), & \text{if } N \text{ is not a perfect square} \\ \frac{1}{2}(a_1 + 1)(a_2 + 1)\ldots(a_k + 1) + 1, & \text{if } N \text{ is a perfect square} \end{cases}$$

(vi) The number of ways in which a composite number N can be resolved into two factors which are relatively prime to each other is equal to 2^{n-1}, where n is the number of different factors in N.

Some Results Based on Geometrical Problems

(i) If n distinct points are given in the plane such that no three of which are collinear, then the number of line segments formed = ${}^nC_2$ and if m of these points are collinear $(m \geq 3)$, then number of line segments is $({}^nC_2 - {}^mC_2 + 1)$.

(ii) The number of diagonals in an n sided closed polygon $= {}^nC_2 - n$.

(iii) If n distinct points are given in the plane such that no three of which are collinear, then number of triangles formed = ${}^nC_3$

If m of these points are collinear $(m \geq 3)$, then number of triangles formed = ${}^nC_3 - {}^mC_3$

(iv) If n distinct points are given on the circumference of circle, then

(a) Number of straight lines = ${}^nC_2$

(b) Number of triangles = ${}^nC_3$

(c) Number of quadrilaterals = ${}^nC_4$

(d) Number of k sided polygons = ${}^nC_k$

(v) In a square of size $n \times n$, number of rectangles of any size is equal to $\sum_{r=1}^{n} r^3$ and number of squares of any size is $\sum_{r=1}^{n} r^2$.

(vi) Number of rectangles of any size in a rectangle of size $n \times p\,(n < p)$ is $\frac{np}{4}(n+1)(p+1)$ and number of squares of any size is $\sum_{r=1}^{n}(n+1-r)(p+1-r)$.

(vii) If m parallel vertical lines are intersected by n parallel horizontal lines, then number of parallelograms, so formed is ${}^mC_2 \times {}^nC_2 = \frac{mn(m-1)(n-1)}{4}$.

Derangement

Any change in the given order of things is called a derangement.

(i) If n things form an arrangement in a row, then the number of ways in which they can be derange, so that none of them occupies its original place, is D_n

$$D_n = n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ldots + (-1)^n \frac{1}{n!}\right)$$

(ii) D_n = Number of ways, if all n things goes to wrong place. Thus, D_r = Number of ways, if r things goes to wrong place. If r goes to wrong places out of n, then $(n-r)$ goes to correct places $= {}^nC_{n-r}D_r$

where, $D_r = r!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ldots + (-1)^r \frac{1}{r!}\right]$

Best Practice SHOTS

25. From 12 books, the difference between the number of ways a selection of 5 books when one specified book is always excluded and one specified book is always included is

(a) 64 (b) 118
(c) 132 (d) 330

26. From 6 different novels and three different dictionaries, 4 novels and 1 dictionary are to be selected and arranged in a row on the self, so that the dictionary is always in the middle. Then, the number of such arrangements is [AIEEE 2009]

(a) atleast 500 but less than 750
(b) atleast 750 but less than 1000
(c) atleast 1000
(d) less than 500

27. 7 relatives of a man comprises 4 ladies and 3 gentlemen and his wife has also 7 relatives, 3 of them are ladies and 4 gentlemen. In how many ways can they invite a dinner party of 3 ladies and 3 gentlemen, so that there are three of man's relative and 3 of the wife's relative?

(a) 485 (b) 500
(c) 486 (d) 102

28. Everybody in a room shakes hands with everybody else. If total number of hand shakes is 66, then total number of persons in the room is

(a) 9 (b) 12
(c) 10 (d) 14

29. A father with 8 children takes 3 at a time to the zoological garden, as often as he can without taking the same 3 children together more than once. The number of times he will go to the garden, is

(a) 112 (b) 56
(c) 336 (d) None of these

30. In a village panchayat election, the number of contestant is one more than the number of maximum candidates for which a voter can vote. If the total number of ways in which a voter can vote is 126, then the number of contestants is

(a) 4 (b) 5 (c) 6 (d) 7

31. The number of positive odd divisors of 216 is

(a) 4 (b) 6 (c) 8 (d) 12

32. In a chess tournament where the participants were to plays one game with one another, two players fell ill having played 6 games each, without playing among themselves. If the total number of games is 117, then the number of participants at the beginning was

(a) 15 (b) 16 (c) 17 (d) 18

33. Let l_1 and l_2 be two lines intersecting at P. If A_1, B_1, C_1 are points on l_1 and A_2, B_2, C_2, D_2, E_2 are points on l_2 and if none of these coincides with P, then the number of triangles formed by these eight points, is

(a) 56 (b) 55 (c) 46 (d) 45

34. Let T_n denote the number of triangles which can be formed using the vertices of a regular polygon of n sides. If $T_{n+1} - T_n = 21$, then n equals [IIT 2001]

(a) 5 (b) 7 (c) 6 (d) 4

35. Six 'X' have to be placed in the squares of the figure given, such that each row contains atleast one X, this can be done in

(a) 24 ways (b) 28 ways (c) 26 ways (d) 36 ways

36. The number of ways in which a mixed double game can be arranged from amongst 9 married couples, if no husband and wife play in the same game, is

(a) 756 (b) 1512
(c) 3024 (d) None of these

37. There are 10 points in a plane of which no three points are collinear and four points are concyclic. The number of different circles that can be drawn through atleast 3 points from the given points, is

(a) 116 (b) 117
(c) 120 (d) None of these

38. The number of odd proper divisors and even proper divisors of 5040 are respectively

(a) 40, 10 (b) 11, 47
(c) 12, 39 (d) None of these

39. There are 10 girls and 8 boys in a classroom including Mr Ravi, Ms Rani and Ms Radha. A list of speakers consisting of 8 girls and 6 boys has to be prepared. Mr Ravi refuses to speak, if Ms Rani is speaker. Ms Rani is refuses to speak, if Ms Radha is speaker. The number of ways the list can be prepared, is

(a) 202 (b) 308 (c) 567 (d) 952

40. There are two urns. Urn A has 3 distinct red balls and urn B has 9 distinct blue balls. From each urn two balls are taken out at random and then transferred to the other. The number of ways in which this can be done, is [AIEEE 2010]

(a) 66 (b) 108 (c) 3 (d) 36

41. Abhishika has 10 friends among whom two are married to each other. She wishes to invite five of them for a party. If the married couples refuse to attend separately, then the number of different ways in which she can invite five friends, is

(a) ${}^8C_5$ (b) $2 \times {}^8C_3$
(c) ${}^{10}C_5 - 2 \times {}^8C_5$ (d) None of these

42. $ABCD$ is a convex quadrilateral and 3, 4, 5 and 6 points are marked on the sides AB, BC, CD and DA, respectively. The number of triangles with vertices on different sides is

(a) 270 (b) 220 (c) 282 (d) 342

43. A person predicts the outcome of 20 cricket matches of his home team. Each match can result in either win, loss or tie for the home team. Total number of ways in which he can make the predictions, so that exactly 10 predictions are correct, is equal to

(a) ${}^{20}C_{10} \times 2^{10}$ (b) ${}^{20}C_{10} \times 3^{20}$

(c) ${}^{20}C_{10} \times 3^{10}$ (d) ${}^{20}C_{10} \times 2^{20}$

44. A candidate is required to answer six out of 10 questions which are divided into two groups each containing five questions. He is not permitted to attempt more than four questions from either group. The number of different ways in which the candidate can choose six questions, is

(a) 50 (b) 150

(c) 200 (d) 250

45. The total number of positive solutions of $15 < x_1 + x_2 + x_3 \leq 20$ is

(a) 685 (b) 785 (c) 1125 (d) None

46. The number of triangles whose vertices are at the vertices of an octagon but none of whose sides happen to come from the sides of the octagon is

(a) 24 (b) 52 (c) 48 (d) 16

47. The total number of 5-digit numbers of different digits in which the digit in the middle is the largest, is

(a) $30 \times 3!$ (b) $33 \times 3!$ (c) $\sum_{n=4}^{9} {}^{n}C_4 \times 4!$ (d) None

48. Let A be the set of all positive prime integers less than 30. The number of different rational numbers whose numerator and denominator belong to A, is

(a) 90 (b) 180 (c) 91 (d) None

49. The number of integer solutions for the equation $x + y + z + t = 20$, where x, y, z, t are all greater than or equal to -1, is

(a) ${}^{20}C_4$ (b) ${}^{23}C_3$ (c) ${}^{27}C_4$ (d) ${}^{27}C_3$

50. There are four balls of different colours and four boxes of colours same as those of the balls. Then, the number of ways in which the balls, one in each box could be placed in such a way that a ball does not go to the box of its own colour, is

(a) 10 (b) 9 (c) 8 (d) 7

Answers

1. (c)	**2.** (d)	**3.** (a)	**4.** (a)	**5.** (c)	**6.** (a)	**7.** (d)	**8.** (c)	**9.** (b)	**10.** (b)
11. (d)	**12.** (c)	**13.** (c)	**14.** (a)	**15.** (b)	**16.** (d)	**17.** (c)	**18.** (a)	**19.** (b)	**20.** (a)
21. (a)	**22.** (c)	**23.** (a)	**24.** (c)	**25.** (c)	**26.** (c)	**27.** (a)	**28.** (b)	**29.** (b)	**30.** (d)
31. (a)	**32.** (a)	**33.** (d)	**34.** (b)	**35.** (c)	**36.** (c)	**37.** (b)	**38.** (b)	**39.** (b)	**40.** (b)
41. (b)	**42.** (d)	**43.** (a)	**44.** (c)	**45.** (a)	**46.** (d)	**47.** (d)	**48.** (a)	**49.** (d)	**50.** (b)

CBSE CHAPTERWISE
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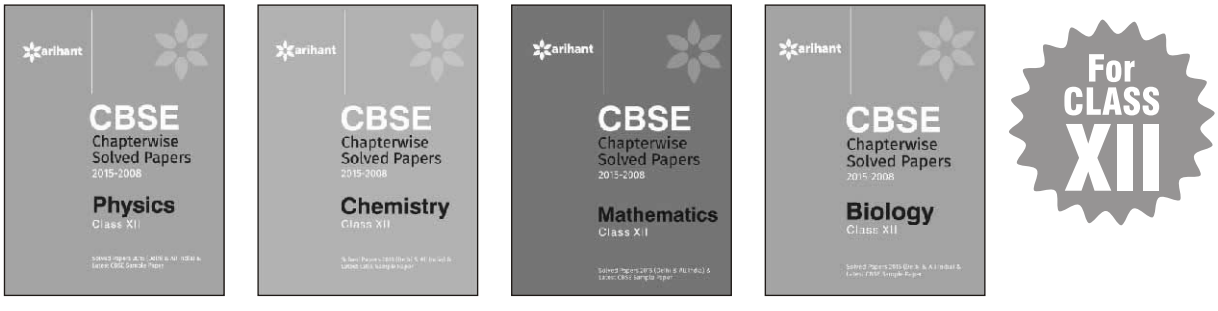

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Formula at a Glance

1. ${}^nP_0 = 1,\ {}^nP_r = n \cdot {}^{n-1}P_{r-1}$

2. Number of permutations of n dissimilar things taken r at a time when p particular things always occur is ${}^{n-p}C_{r-p} \cdot r!$ never occur is ${}^{n-p}C_r$.

3. The number of permutations of n different things taken r at a time when each thing may be repeated any number of times is n^r.

4. The number of permutations of n things of which p are alike of one kind, q are alike of second kind, r are alike of third kind and remaining are distinct, is $\frac{n!}{p!q!r!}$.

5. Number of permutations of n different things, taken all at a time, when m specified things
   always come together is $m! \times (n - m + 1)!$.
   never come together is $n! - m! \times (n - m + 1)!$.

6. The number of ways in which n persons can be seated in a round table is $(n - 1)!$ (If clockwise and anti-clockwise) orders are taken as different)

7. The number of ways in which n different beads can be arranged to form a necklace, is $\frac{(n-1)!}{2}$.
   (If clockwise and anti-clockwise orders are taken as not different).

8. The number of combinations of n different things taken r at a time, where p particular things
   (i) always occur is ${}^{n-p}C_{r-p}$.
   (ii) never occur is ${}^{n-p}C_r$.

9. The number of ways of n different things selecting atleast one of them is ${}^nC_1 + {}^nC_2 + \ldots + {}^nC_n = 2^n - 1$.

10. (i) The number of ways in which $(m + n)$ different things can be divided into two groups which contain m and n things respectively is
    $${}^{m+n}C_m \cdot {}^nC_n = \frac{(m+n)!}{m!n!},\ m \neq n.$$
    (ii) If order of group is not important, then number of ways in which $2n$ different things can be divided equally into two groups is $\frac{(2n)!}{2!(n!)^2}$.
    (iii) If order of group is important, then number of ways in which $2n$ different things can be divided equally into two distinct groups is $\frac{2n!}{(n!)^2}$.

11. The number of ways in which $(m + n + p)$ different things can be divided into three groups which contain m, n wp things respectively, is $\frac{(m+n+p)!}{m!n!p!},\ m \neq n \neq p$.

12. If out of $(p + q + r + k)$, p are alike of one kind, q are alike of second kind, r are alike of third kind and t are different, then total number of combinations is $(p + 1)(q + 1)(r + 1)2^k - 1$.

13. If $N = p_1^{a_1} \cdot p_2^{a_2} \cdot p_3^{a_3} \ldots p_k^{a_k}$, where $p_1, p_2, p_3, \ldots, p_k$ are different primes and $a_1, a_2, a_3, \ldots, a_k$ are natural numbers, then total number of divisors of N
    (i) including 1 and $N = (a_1 + 1)(a_2 + 1)(a_3 + 1)\ldots(a_k + 1)$
    (ii) excluding 1 and $N = (a_1 + 1)(a_2 + 1)(a_3 + 1)\ldots(a_k + 1) - 2$
    (iii) excluding 1 or $N = (a_1 + 1)(a_2 + 1)(a_3 + 1)\ldots(a_k + 1) - 1$

14. The number of ways in which n different things can be arranged into r different groups is ${}^{n+r-1}P_r$ or $n! \cdot {}^{n-1}C_{r-1}$ according as blank group are or are not admissible.

15. The number of ways in which n different things can be distributed into r different groups is
    $$r^n - {}^rC_1 (r-1)^n + {}^rC_2 (r-2)^n - \ldots + (-1)^{r-1} {}^rC_{r-1}$$

16. The total number of non-negative integral solutions of the equation $x_1 + x_2 + \ldots + x_n = n$ is ${}^{n+r-1}C_{r-1}$ and total number of solutions in the set N of natural numbers is ${}^{n-1}C_{r-1}$.

17. (i) The sum of the digits in the unit place of all numbers formed with the help of $a_1, a_2, \ldots, a_n$ taken all at a time is $(n - 1)!(a_1 + a_2 + \ldots + a_n)$ (repetition of digits is not allowed)
    (ii) The sum of all digits numbers that can be formed using the digits $a_1, a_2, \ldots, a_n$ is $(n - 1)!(a_1 + a_2 + \ldots + a_n)\left(\frac{10^n - 1}{9}\right)$
    (repetition of digits is not allowed).

18. If in any party n persons are present, then total number of hand shakes $= {}^nC_2$.

19. (i) The number of triangles formed by joining n points is ${}^nC_3$.
    (ii) The number of triangles formed by joining n points out of which m are collinear, is ${}^nC_3 - {}^mC_3$.

20. The number of straight lines passing through n points out of which m are collinear $= {}^nC_2 - {}^mC_2 + 1$.

21. The number of diagonals in a polygon of n sides is ${}^nC_2 - n$ or $\frac{n(n-3)}{2}$.

22. If m parallel lines in a plane are intersected by a family of other n parallel lines. Then, total number of parallelogram, so formed is ${}^mC_2 \times {}^nC_2$ or $\frac{mn(m-1)(n-1)}{4}$.

23. If n distinct objects are arranged in a row, then the number of ways in which they can be dearranged, so that none of them occupies its original place, is
    $$n!\left\{1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \ldots + (-1)^n \frac{1}{n!}\right\}.$$

Probability

Probability is defined as a quantitative measure of uncertainty. It is an approximate tool for many diverse field. Probability is very important topic for JEE Main and Advanced point of view. Last year atleast one question in JEE Main and three questions in JEE Advanced were asked. A good understanding of permutation and combination will help you to great extent.

PROBABILITY OF AN EVENT, ADDITION THEOREM

Terminology used in Probability

Sample Space

The set of all possible outcomes of a trial(random experiment) is called its sample space.

In general, it is denoted by S and outcomes of trial are known as sample points.

Event

The subset of sample space is called an event.

Note

(i) Sample space S plays the same role as the universal set for all problems related to the particular experiment.
(ii) ϕ is also a subset of S which is called an impossible event.
(iii) S is also a subset of S which is called a sure event or a certain event.

Types of Event

Simple Event

An event containing only a single sample point is called simple event or elementary event.

Compound Event

Event obtained by combining two or more elementary events is called compound event.

Equally Likely Events

Outcomes are said to be equally likely, when we have no reason to believe that one is more likely to occur than the other.

Mutually Exclusive

Events are said to be mutually exclusive or disjoint, if occurrence of any one of them prevents the occurrence of all the others.

Independent Events

Events are said to be independent, if the happening (or non-happening) of one event is not affected by the happening (or non-happening) of others.

Dependent Events

Two or more events are said to be dependent, if the happening of one event affects (partially or totally) the happening of other events.

Exhaustive Events

If $E_1, E_2, E_3 \ldots, E_n$ are n events of a sample S and if

$$S = E_1 \cup E_2 \cup E_3 \cup \ldots \cup E_n = \bigcup_{i=1}^{n} E_i, \text{ then}$$

$E_1, E_2, E_3, \ldots, E_n$ are called exhaustive events.

In other words, events $E_1, E_2, \ldots, E_n$ are said to be exhaustive, if atleast one of them necessarily occurs when ever the experiment is performed.

Mutually Exclusive and Exhaustive Events

A set of events is said to be mutually exclusive and exhaustive, if following conditions are satisfied.

(i) $E_1 \cup E_2 \cup E_3 \cup E_4 \cup \ldots \cup E_n = S$
(ii) $E_1 \cap E_2 \cap E_3 \cap E_4 \ldots \cap E_n = \phi$

Note The collection of elementary events associated with a random experiment always form a system of mutually exclusive and exhaustive system of experiments. In this system

$$P(E_1 \cup E_2 \cup E_3 \cup \ldots \cup E_n)$$
$$= P(E_1) + P(E_2) + P(E_3) + \ldots + P(E_n) = 1$$

Axiomatic Approach of Probability

In this approach, some axioms or rules are depicted to assign probabilities.

Let $S = \{w_1, w_2, w_3, \ldots, w_n\}$ be the sample space associated with a random experiment. Then, a function P which assigns every event $E \subset S$ to a unique non-negative real number $P(E)$ is called probability function, if the following axioms hold

(i) $0 \le P(w_i) \le 1$ for all $w_i \in S$

(ii) $P(S) = 1$, i.e.

$P(w_1) + P(w_2) + \ldots + P(w_n) = 1$

(iii) For any event $E \subset S$,

$P(E) = \sum_{w_i \in A} P(w_i)$, the number $P(w_i)$ is called the probability of elementary event w_i.

Classical Definition of Probability

Let S be the sample space, then probability of occurrence of an event E is denoted by $P(E)$ and is defined as

$$P(E) = \frac{n(E)}{n(S)} = \frac{\text{Number of elements in } E}{\text{Number of elements in } S}$$

$$= \frac{\text{Number of cases favourable to } E}{\text{Total number of cases}}$$

REMEMBER

(i) $0 \le P(E) < 1$

(ii) $P(E) + P(\bar{E}) = 1$

(iii) $P(A) \subseteq P(B)$ stands for the occurrence of A implies occurrence of B.

(iv) $P(\bar{A})$ or $P(A')$ stands for the non-occurrence or negation of A.

(v) $P(A \cup B)$ stands for the occurrence of atleast one of A and B.

(vi) $P(A \cap B)$ stands for simultaneous occurrence of A and B.

(vii) $P(A' \cap B')$ stands for non-occurrence of both A and B.

Odds in Favour and Odds against of an Event

Let S be the sample space and E be an event. Let E' denote the complement of event E. Then,

(i) odds in favour of event

$$(E) = \frac{\text{Number of cases favourable to } E}{\text{Number of cases against to } E}$$

$$= \frac{n(E)}{n(E')} = \frac{n(E)/n(S)}{n(E')/n(S)} = \frac{P(E)}{P(E')}$$

(ii) odds against of an event (E)

$$= \frac{\text{Number of cases against to } E}{\text{Number of cases favourable to } E}$$

$$= \frac{n(E')}{n(E)} = \frac{n(E')/n(S)}{n(E)/n(S)} = \frac{P(E')}{P(E)}$$

Playing Cards

A standard deck of 52 cards has four suits : hearts (♥), clubs (♣), diamonds (♦) and spades (♠), with 13 cards of each suit. Each suit has a jack, a queen and a king (sometimes called a face cards), an ace and cards numbered from 2 to 10. The hearts and the diamonds are red and the spades and the clubs are black.

Addition Theorem of Probability

If A and B are any two events in a sample space S, then the probability of occurrence of atleast one of the events A and B is given by

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

In general, $P(A_1 \cup A_2 \cup \ldots \cup A_n) = \sum_{i=1}^{n} P(A_i)$

$$- \sum_{i<j} P(A_i \cap A_j) + \sum_{i<j<k} P(A_i \cap A_j \cap A_k)$$

$$- \ldots + (-1)^{n-1} P(A_1 \cap A_2 \cap A_3 \cap \ldots \cap A_n)$$

Note For mutually exclusive events

(a) $P(A \cup B) = P(A) + P(B)$

(b) $P(A \cup B \cup C) = P(A) + P(B) + P(C)$

REMEMBER

(i) If A, B and C are three events in a sample space S, then

$P(A \cup B \cup C) = P(A) + P(B) + P(C) -$

$P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$

(ii) If A and B are any two events, then

(a) $P(A) = P(A - B) + P(A \cap B) = P(A \cap B') + P(A \cap B)$

(b) $P(B) = P(B - A) + P(A \cap B) = P(B \cap A') + P(A \cap B)$

(iii) P (Atleast two of A, B and C occur)

$= P(A \cap B) + P(B \cap C) + P(C \cap A) - 2P(A \cap B \cap C)$

(iv) P (Exactly two of A, B and C occur)

$= P(B \cap C) + P(C \cap A) + P(A \cap B) - 3P(A \cap B \cap C)$

(v) P (Exactly one of A, B and C occurs)

$= P(A) + P(B) + P(C) - 2P(B \cap C) - 2P(C \cap A)$

$- 2P(A \cap B) + 3P(A \cap B \cap C)$

SPEED BOOSTER

(i) When three dice are thrown, let n $(3 \le n \le 18)$ be the sum of numbers on them, then number of ways of getting the sum n is given by

$$\begin{cases} \frac{(n-1)(n-2)}{2}, & \text{if } 3 \le n \le 8 \\ \frac{(19-n)(20-n)}{2}, & \text{if } 13 \le n \le 18 \\ 25, & \text{if } (k = 9, 12) \\ 27, & \text{if } (k = 10, 11) \end{cases}$$

(ii) If there are n letters corresponding to n envelopes are placed in the envelopes at random, then

(a) probability that all letters are in right envelopes $= \frac{1}{n!}$

(b) probability that all letters are not in right envelopes

$$= 1 - \frac{1}{n!}$$

(c) probability that no letter is in right envelopes

$$= \left(\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \ldots + (-1)^n \frac{1}{n!}\right)$$

(d) probability that exactly r letters are in right envelopes

$$= \frac{1}{r!}\left[\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \ldots + (-1)^{n-r} \frac{1}{(n-r)!}\right]$$

(iii) Selection of shoes from a cupboard

(a) Out of n pair of shoes, if k shoes are selected at random, then the probability that there is no pair, is $\frac{{}^{n}C_{k}\, 2^{k}}{{}^{2n}C_{k}}$.

(b) The probability that there is atleast one pair, is $(1 - p)$.

(iv) **Selection of squares from the chessboard** If r squares are selected from a chessboard, then probability that they lie on a diagonal is

$$\frac{4[{}^{7}C_r + {}^{6}C_r + {}^{5}C_r + \ldots + {}^{1}C_r] + 2 \cdot {}^{8}C_r}{{}^{64}C_r}$$

(v) Let A be a set containing n elements. Then, probability that

(a) r subset of A, selected at random have no common elements is $\left(\frac{2^r - 1}{2^r}\right)^n$.

(b) Two subsets of A selected at random will have r elements in common is

$${}^{n}C_r \frac{3^{n-r}}{4^n}.$$

(c) If B and C are two subsets of A, then probability that $C \cup B = A$ is $\left(\frac{3}{4}\right)^m$.

(vi) If A and B are two finite sets and if a mapping is selected at random from the set of all mapping from A into B, then the probability that the mapping is

(a) a one-one function $= \frac{{}^{n(B)}P_{n(A)}}{n(B)^{n(A)}}$

(b) a many-one function $= 1 - \frac{{}^{n(B)}P_{n(A)}}{n(B)^{n(A)}}$

(c) a constant function $= \frac{n(B)}{n(B)^{n(A)}}$

(d) a one-one and onto function $= \frac{n(A)!}{n(B)^{n(A)}}$

Best Practice SHOTS

1. Three houses are available in a locality. Three persons apply for the houses. Each applies for one house without consulting others. The probability that all the three apply for the same house, is **[AIEEE 2005]**

(a) $\frac{7}{9}$ (b) $\frac{8}{9}$ (c) $\frac{1}{9}$ (d) $\frac{2}{9}$

2. Two distinct numbers are chosen at random from the first 15 natural numbers. The probability that the sum will be divisible by 3, is

(a) $\frac{30}{91}$ (b) $\frac{31}{91}$

(c) $\frac{60}{91}$ (d) None of these

3. A five digit number is formed by the digits 1, 2, 3, 4, 5, 6 and 8. The probability that the number has even digit at both ends, is

(a) $\frac{3}{7}$ (b) $\frac{4}{7}$ (c) $\frac{2}{7}$ (d) None of these

4. Number 1, 2, 3,..., 100 are written down on each of the cards A, B and C. One number is selected at random from each of the cards. The probability that the numbers so selected can be the measures (in cm) of three sides of a right angled triangle, is

(a) $\frac{4}{100^3}$ (b) $\frac{3}{50^3}$

(c) $\frac{3!}{100^3}$ (d) None of these

5. If three distinct numbers are chosen randomly from the first 100 natural numbers, then the probability that all three of them are divisible by both 2 and 3, is **[IIT 2004]**

(a) $\frac{4}{25}$ (b) $\frac{4}{35}$ (c) $\frac{4}{33}$ (d) $\frac{4}{1155}$

6. 10 mangoes are to be distributed among 5 persons. The probability that atleast one of them will receive none, is

(a) $\frac{35}{143}$ (b) $\frac{108}{143}$

(c) $\frac{18}{143}$ (d) $\frac{125}{143}$

7. There is a five volume dictionary among 50 books arranged on a self in a random order. If the volumes are not necessarily kept side-by-side, then the probability that they occur in increasing order from left to right, is

(a) $\frac{1}{5}$ (b) $\frac{1}{5^{50}}$

(c) $\frac{1}{50^5}$ (d) None of these

8. There are four machines and it is known that exactly two of them are faulty. They are tested, one-by-one in a random order fills both the faulty machines are identified. Then, the probability that only two tests are needed, is **[IIT 1998]**

(a) $\frac{1}{3}$ (b) $\frac{1}{6}$ (c) $\frac{1}{2}$ (d) $\frac{1}{4}$

9. In a party 23 persons take their sheets at around table. The odds against two particular person sitting together are

(a) 10 : 1 (b) 1 : 11

(c) 9 : 10 (d) None of these

10. In a convex hexagon two diagonals are drawn at random. The probability that the diagonals intersect at an interior point of the hexagon is

(a) $\frac{5}{12}$ (b) $\frac{7}{12}$

(c) $\frac{2}{5}$ (d) None of these

11. A car is parked by an owner amongst 25 cars in a row, not at either end. On his return he find that exactly 15 places are still occupied. The probability that both the neighbouring places are empty, is

(a) $\frac{91}{276}$ (b) $\frac{15}{184}$

(c) $\frac{15}{92}$ (d) None of these

12. Six faces of a die are marked with the numbers 1, –1, 0 –2, 2 and 3. The die is thrown thrice. The probability that the sum of the numbers thrown is six, is

(a) $\frac{1}{72}$ (b) $\frac{1}{12}$

(c) $\frac{5}{108}$ (d) $\frac{1}{36}$

13. A and B play a game where each is asked to select a number from 1 to 25. If the two numbers match, both of them win a prize. The probability that they will not win a prize in a single trial, is

(a) $\frac{1}{25}$ (b) $\frac{24}{25}$

(c) $\frac{2}{25}$ (d) None of these

14. Let ω be a complex cube root of unity with $\omega \neq 1$. A fair die is thrown three times. If r_1, r_2 and r_3 are the numbers obtained on the die, then the probability that $\omega^{r_1} + \omega^{r_2} + \omega^{r_3} = 0$ is **[IIT 2010]**

(a) $\frac{1}{18}$ (b) $\frac{1}{9}$ (c) $\frac{2}{9}$ (d) $\frac{1}{36}$

15. If four dice are thrown together. Probability that the sum of the number appearing on them is 13, is

(a) $\frac{35}{324}$ (b) $\frac{5}{216}$

(c) $\frac{11}{216}$ (d) $\frac{11}{432}$

16. A is a set containing n elements. A subset P of A is chosen at random. The set A is reconstructed by replacing the element of P. A subset Q is again chosen at random. The probability that $P = Q$ is

(a) $\frac{1}{2^n}$ (b) $\left(\frac{3}{4}\right)^n$ (c) $\left(\frac{1}{4}\right)^n$ (d) $\left(\frac{2}{3}\right)^n$

17. A is a set containing n elements. A subset P of A is chosen at random. The set A is reconstructed by replacing the element of P. A subset Q is again chosen at random. The probability that P and Q are disjoint sets, is

(a) $\left(\frac{1}{2}\right)^n$ (b) $\left(\frac{1}{4}\right)^n$ (c) $\left(\frac{3}{4}\right)$ (d) $\left(\frac{3}{4}\right)^n$

18. Four numbers are multiplied together. Then, the probability that the product will be divisible by 5 or 10 is

(a) $\frac{369}{625}$ (b) $\frac{399}{625}$ (c) $\frac{123}{625}$ (d) $\frac{133}{625}$

19. A and B are two events such that odds against A are 2:1 odds in favour of $A \cup B$ are 3 : 1. If $x \leq P(B) \leq y$, then the ordered pair (x, y) is

(a) $\left(\frac{5}{12}, \frac{3}{4}\right)$ (b) $\left(\frac{2}{3}, \frac{3}{4}\right)$

(c) $\left(\frac{1}{3}, \frac{3}{4}\right)$ (d) None of these

20. Events A, B and C are mutually exclusive events such that $P(A) = \frac{3x+1}{3}$, $P(B) = \frac{1-x}{4}$ and $P(C) = \frac{1-2x}{2}$. The set of all possible values of x is in the interval **[AIEEE 2003]**

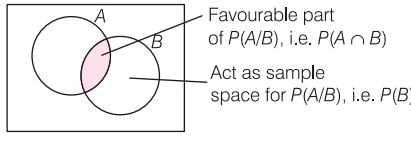
(a) [0, 1] (b) $\left[-\frac{1}{3}, \frac{1}{2}\right]$

(c) $\left[\frac{1}{3}, \frac{2}{3}\right]$ (d) $\left[\frac{1}{3}, \frac{13}{3}\right]$

CONDITIONAL PROBABILITY, BAYE'S THEOREM

Conditional Probability

Let A and B be any two events, $B \neq \phi$, then $P\left(\frac{A}{B}\right)$ denotes the conditional probability of occurrence of event A when B has already occurred. $P\left(\frac{A}{B}\right) = \frac{P(A \cap B)}{P(B)}$



While finding $P\left(\frac{A}{B}\right)$, B work as sample space and $(A \cap B)$ work as the favourable events as shown in figure above.

Multiplication Theorem

If A and B are two events, then $P(A \cap B) = P(B) \cdot P\left(\frac{A}{B}\right)$, where $B \neq \phi$ and $P\left(\frac{A}{B}\right)$ denotes probability of occurrence of A when B has already occurred.

Also, if A, B and C are three events, then

$$P(A\cap B\cap C)=P(A)\cdot P\left(\frac{B}{A}\right)\cdot P\left(\frac{C}{A\cap B}\right)$$

In general $P\left(\bigcap_{i=1}^{n}A_i\right)=P(A_1)\cdot P\left(\frac{A_2}{A_1}\right)\cdot$

$$P\left(\frac{A_3}{A_1\cap A_2}\right)\ldots P\left(\frac{A_n}{A_1\cap A_2\cap\ldots\cap A_{n-1}}\right)$$

REMEMBER

(i) If two events are independent, then
$P(A\cap B)=P(A)\cdot P(B)$

(ii) If A, B and C are three independent events, then
$P(A\cap B\cap C)=P(A)\cdot P(B)\cdot P(C)$

(iii) If A and B are independent events, then
$P(A\cup B)=1-P(A')P(B')$

(iv) If A and B are independent events, then
(a) A and B' are independent events.
(b) A' and B are independent events.
(c) A' and B' are independent events.

(v) $P(A/B)+P(\bar{A}/B)=1$

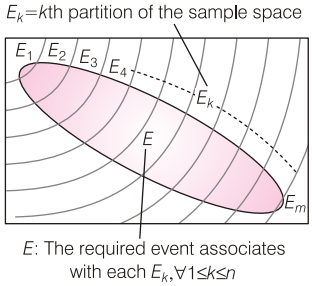
(vi) $P(B)=P(A)\cdot P(B/A)+P(\bar{A})\cdot P\left(\frac{B}{\bar{A}}\right)$

Law of Total Probability

Let S be the sample space and let $E_1,E_2,E_3,\ldots,E_n$ be n mutually exclusive and exhaustive events associated with a random experiment. If E is any event which occurs, if anyone of the events $E_1,E_2,\ldots,E_n$ occurs, then

$$P(E)=P(E_1)\cdot P\left(\frac{E}{E_1}\right)+P(E_2)\cdot P\left(\frac{E}{E_2}\right)+P(E_3)\cdot P\left(\frac{E}{E_3}\right)+\ldots+P(E_n)\cdot P\left(\frac{E}{E_n}\right)$$

or $$P(E)=\sum_{i=1}^{n}P(E_i)\,P\left(\frac{E}{E_i}\right)$$



Partition of a Set

Consider a sample space S. Let $E_1, E_2,\ldots,E_n$ be the set of mutually exclusive and exhaustive events of sample space S. These events are said to be partition of sample space S, if $E_i\cap E_j=\phi$ for $i\neq j, 1\le i, j\le n$ and $\sum_{i=1}^{n}P(E_i)=1$

Baye's Theorem

If $E_1,E_2,E_3,\ldots,E_n$ are n mutually exclusive and exhaustive events and E is an event which occurs together with either of E_i, then

$$P\left(\frac{E_i}{E}\right)=\frac{P(E_i)\cdot P\left(\frac{E}{E_i}\right)}{\sum_{i=1}^{n}P(E_i)P\left(\frac{E}{E_i}\right)},\ i=1,2,3,\ldots,n.$$

Note

(i) If $P(E_1)=P(E_2)=\ldots=P(E_n)$, then by Baye's theorem

$$P\left(\frac{E_k}{E}\right)=\frac{P\left(\frac{E}{E_k}\right)}{P\left(\frac{E}{E_1}\right)+P\left(\frac{E}{E_2}\right)+\ldots+P\left(\frac{E}{E_n}\right)}$$

(ii) If $E_1,E_2,E_3,\ldots,E_n$ form a partition of an event E, then

$$P\left(\frac{E_k}{E}\right)=\frac{P(E_k)}{P(E_1)+P(E_2)+\ldots+P(E_n)}$$

(iii) $P\left(\frac{E}{E_i}\right)$ gives the contribution of E_i in the occurrence of E

(iv) The probabilities $P(E_1)\cdot P(E_2)\ldots P(E_n)$ which are known before the experiment takes place are called a priori probabilities and $P\left(\frac{E_i}{E}\right)$ are called posteriori probabilities.

Concept for Asserting the Truth Statement

If P_1 and P_2 are the probabilities of speaking truth of two independent witnesses A and B, then

(i) P (their combined statement is true)

$$=\frac{P_1P_2}{P_1P_2+(1-P_1)(1-P_2)}$$

Note In this case, it has been assumed that we have no knowledge of the event except the statement made by A and B.

(ii) If P is the probability of happening of the event before their statement, then

P (their combined statement is true)

$$=\frac{PP_1P_2}{PP_1P_2+(1-P)(1-P_1)(1-P_2)}$$

Here, it has been assumed that the statement given by all the independent witness can be given by two ways only, so that if all the witness tell falsehood, they agree in telling the same falsehood. If this is not the case and c is the chance of their coincidence, then

Probability that the statement is true

$$=P_1\,P_2\,P$$

Probability that the statement is false

$$=(1-P_1)(1-P_2)(1-P)\,c$$

Best Practice SHOTS

21. If A and B are two independent events such that $P(\overline{A} \cap B) = \frac{2}{15}$ and $P(A \cap \overline{B}) = \frac{1}{6}$, then $P(B)$ is equal to

(a) $\frac{1}{5}$ or $\frac{4}{5}$ (b) $\frac{1}{6}$ or $\frac{5}{6}$ (c) $\frac{4}{5}$ or $\frac{1}{6}$ (d) $\frac{5}{6}$ or $\frac{1}{5}$

22. Cards are drawn from a pack of 52 playing cards one-by-one. The probability that exactly 10 cards will be drawn before the first ace, is

(a) $\frac{451}{884}$ (b) $\frac{241}{1456}$ (c) $\frac{164}{4165}$ (d) None

23. A basket contains 5 apples and 7 mangoes and another basket contains 4 apples and 8 mangoes. One fruit is picked out from each basket. The probability that the fruits are both apples or both mangoes, is

(a) $\frac{24}{144}$ (b) $\frac{56}{144}$ (c) $\frac{68}{144}$ (d) $\frac{76}{144}$

24. The probability that A speaks truth is $\frac{4}{5}$, while this probability for B is $\frac{3}{4}$. The probability that they contradict each other when asked to speak on a fact is **[AIEEE 2004]**

(a) $\frac{4}{5}$ (b) $\frac{1}{5}$ (c) $\frac{7}{20}$ (d) $\frac{3}{20}$

25. For two events A and B, if $P(A) = P\left(\frac{A}{B}\right) = \frac{1}{4}$ and $P\left(\frac{B}{A}\right) = \frac{1}{2}$, then

(a) A and B are independent events

(b) $P\left(\frac{A'}{B}\right) = \frac{3}{4}$

(c) $P\left(\frac{B'}{A'}\right) = \frac{1}{2}$

(d) All of the above

26. The probability that in a year of 22nd century chosen at random, there will be 53 Sundays, is

(a) $\frac{3}{28}$ (b) $\frac{2}{28}$ (c) $\frac{7}{28}$ (d) $\frac{5}{28}$

27. One Indian and four Australian men and their wives are to be seated randomly around a circular table. The conditional probability that the Indian man is seated adjacent to his wife given that each Australian man is seated adjacent to his wife, is **[IIT 2007]**

(a) 1/2 (b) 1/3 (c) 2/5 (d) 1/5

28. A fair die is tossed repeatedly until a six is obtained. Let X denote the number of tosses required. Then, conditional probability that $X \geq 6$ given $X > 3$ equals

(a) $\frac{125}{216}$ (b) $\frac{25}{216}$ (c) $\frac{5}{36}$ (d) $\frac{25}{36}$

29. A signal which can be green or red with probability 4/5 and 1/5 respectively, is received by station A and then transmitted to station B. The probability of each station receiving the signal correctly is 3/4. If the signal received at station B is green, then the probability that the original signal was green, is **[IIT 2010]**

(a) $\frac{3}{5}$ (b) $\frac{6}{7}$ (c) $\frac{20}{23}$ (d) $\frac{9}{20}$

30. Let E' denote the complement of an event E. Let E, F, G be pairwise independent events such that $P(G) > 0$ and $P(E \cap F \cap G) = 0$. Then, $P\left(E' \cap \frac{F'}{G}\right)$ equals

(a) $P(E') + P(F')$ (b) $P(E') - P(F')$
(c) $P(E') - P(F)$ (d) $P(E) - P(F')$

31. The probability that randomly selected calculator from a store is of brand 'r' is proportional to r, $r = 1, 2, ..., 6$. Further the probability of a calculator of brand 'r' being defective is $\frac{7 - r}{21}$, $r = 1, 2, ..., 6$. Then, the probability that a calculator randomly selected from the store being defective, is **[AIEEE 2012]**

(a) $\frac{8}{23}$ (b) $\frac{13}{63}$ (c) $\frac{55}{63}$ (d) $\frac{50}{63}$

32. Let U_1 and U_2 be two urns such that U_1 contains 3 white and 2 red balls and U_2 contains only 1 white balls. A fair coin is tossed. If head appear, then 1 ball is drawn at random from the urn U_1 and put into U_2. However, if tail appears, then 2 balls are drawn at random from U_1 and put into U_2. Now, 1 ball is drawn at random from U_2. It is given that the ball drawn from U_2 is white, the probability that head appeared on the coin, is

(a) $\frac{17}{23}$ (b) $\frac{11}{23}$ (c) $\frac{15}{23}$ (d) $\frac{12}{23}$

33. An experiment has 10 equally likely outcomes. Let A and B be two non-empty events of the experiment. If A consists of 4 outcomes, then the number of outcomes that B must have so that A and B are independent, is

(a) 2, 4 or 8 (b) 3, 6 or 9 (c) 4 or 8 (d) 5 or 10

34. There are n urn each containing $(n + 1)$ balls such that the ith urn contain 'i' white balls and $(n + 1 - i)$ red balls. Let U_i be the event of selecting ith urn $i = 1, 2, 3, ..., n$ and w denotes the event of getting a white ball. If $P(U_i) \propto i$, where $i = 1, 2, 3, ..., n$, then $\lim_{n \to \infty} P(w)$ is equal to

(a) 1 (b) 2/3 (c) 1/4 (d) 3/4

35. There are two bags one of which contains 3 black and 4 white balls while the other contains 4 black and 3 white balls. A die is cast. If the face 1 or 3 turns up, a ball is taken from the first bag and if any other face turn

up, a ball is chosen from the second bag. Then, the probability of choosing a black ball is

(a) $\frac{10}{21}$ (b) $\frac{11}{21}$ (c) $\frac{12}{23}$ (d) None

36. The probability that certain electronic component fails when first used is 0.10. If it does not fail immediately, the probability that it lasts for one year is 0.99. Then, the probability that a new component will last for one year, is

(a) 0.10 (b) 0.90 (c) 0.891 (d) 0.8

37. A bag contains n balls out of which some balls are white. If probability that a bag contains exactly 'i' white ball is proportional to i^2. A ball is drawn at random from the bag and found to be white, then the probability that bag contains exactly two white balls, is

(a) $\frac{72}{n^2}$ (b) $\frac{48}{n^2}$ (c) $\frac{24}{n^2}$ (d) $\frac{72}{n^2(n+1)}$

38. The chances of defective screw in three boxes A, B, C are $\frac{1}{5}, \frac{1}{6}, \frac{1}{7}$, respectively. A box is selected at random and a screw drawn from it at random is found to be defective. Then, the probability that it came from box A is

(a) $\frac{41}{107}$ (b) $\frac{42}{107}$ (c) $\frac{36}{107}$ (d) $\frac{72}{107}$

39. Each of n urns contains 4 white and 6 black balls. The $(n+1)$th urn contains 5 white and 5 black balls. One of the $(n+1)$ urns is chosen at random and two balls are drawn from it without replacement. Both the balls turn out to be black. If the probability that the $(n+1)$th urn was chosen to draw the balls is 1/16, then the value of 'n' is

(a) 9 (b) 8 (c) 7 (d) 10

40. A real estate man has eight master keys to open several new homes. Only one master key will open any given house. If 40% of these homes are usually left unlocked, then the probability that the real estate man can get into a specific home, if he select three master keys at random before leaving the office, is

(a) $\frac{7}{8}$ (b) $\frac{6}{5}$ (c) $\frac{3}{4}$ (d) $\frac{5}{8}$

PROBABILITY DISTRIBUTION

Random Variable

Let S be a sample space associated with a given random experiment. A real-valued function X defined on S as $X : S \to R$ is called a random variable.

There are two types of random variable.

Discrete Random Variable

If the range of the real function $X : S \to R$ is a finite set or an infinite set of real numbers, then it is called a discrete random variable.

Continuous Random Variable

If the range of X is an interval (a, b) of R, then X is called a continuous random variable.

Probability Distribution

If the random variable X takes n distinct values $x_1, x_2, x_3, \ldots, x_n$, then $\{X = x_1\}, \{X = x_2\}, \{X = x_3\}, \ldots, \{X = x_n\}$ are mutually exclusive and exhaustive events. Then system of distribution of numbers

X	x_1 x_2 x_3 … x_n
$P(X)$	P_1 P_2 P_3 … p_n

is said to be the probability distribution of the random variable X.

The expectation (mean), variance and standard deviation of random X is given below

(i) The expectation (mean) of random variable X is $E(X) = \sum_{i=1}^{n} x_i p_i$

(ii) **Variance** $\text{Var}(X) = \sigma_x^2 = E(X^2) - \{E(X)\}^2$

where, $E(X^2) = \sum_{i=1}^{n} x_i^2 p_i$

(iii) **Standard deviation** $\sqrt{V(X)} = \sigma_x = \sqrt{E(X^2) - \{E(X)\}^2}$

Note (i) If $Y = aX + b$, then

(a) $E(Y) = E(aX + b) = aE(X) + b$

(b) $\sigma_y^2 = V(Y) = a^2 V(X) = a^2 \sigma_x^2$

(c) $\sigma_y = \sqrt{V(Y)} = |a| \sigma_x$

(ii) If $Z = aX^2 + bX + c$, then

$E(Z) = E(aX^2 + bX + c) = aE(X^2) + bE(X) + c$

Binomial Distribution

Bernaulli Trial

In a random experiment, if there are any two events, "success and failure" and the sum of probabilities of these two events is 1. Then, outcomes of such experiment is known as Bernaulli trial.

Binomial Distribution

The probability of getting r success in n trial of a random experiment is ${}^nC_r p^r q^{n-r}$, where p is probability of success and q is probability of failure in each trial, also $p + q = 1$.

Note

(i) If $p = q$, then probability of r successes in n trial is ${}^nC_r p^n$.

(ii) If the total number of trials is n in any attempt and if there are N such attempts, then the total number of r successes is $N({}^nC_r p^r q^{n-r})$.

(iii) Mean $= E(X) = np$

(iv) Variance $= \sigma_x^2 = npq$

(v) Standard deviation $= \sigma_x = \sqrt{npq}$

(vi) Mean is always greater than variance.

SPEED BOOSTER

(i) If a die has m faces marked with the numbers 1, 2, 3, ..., m and if such n dice are thrown, then the probability that the sum of the numbers exhibited on the upper face equal to p, is given by the coefficient of x^p in the expansion of $\frac{(x + x^2 + x^3 + \ldots + x^m)^n}{m^n}$.

(ii) Let X and Y be random variables on the sample space S and K is a real number, then

(a) $E(KX) = K \cdot E(X)$

(b) $E(X + Y) = E(X) + E(Y)$

Geometrical Probability

If the total number of outcomes of a trial in a random experiment is infinite, in such cases, the definition of probability is modified and the general expression for the probability of occurrence of an event is given by

$$P(E) = \frac{\text{Measures of specified part of the region}}{\text{Measure of whole region}}$$

where, 'measure' means length, area or volume of the region, if we are dealing with one, two or three dimensional space respectively.

Best Practice SHOTS

41. In a binomial distribution $B\left(n, p = \frac{1}{4}\right)$, if the probability of atleast one success is greater than or equal to $\frac{9}{10}$, then n is greater than

(a) $\frac{1}{\log_{10} 4 - \log_{10} 3}$ (b) $\frac{1}{\log_{10} 4 + \log_{10} 3}$

(c) $\frac{9}{\log_{10} 4 - \log_{10} 3}$ (d) $\frac{4}{\log_{10} 4 - \log_{10} 3}$

42. The mean and variance of a binomial variable X are 2 and 1, respectively. The probability that X takes values greater than 1, is

(a) $\frac{5}{16}$ (b) $\frac{8}{16}$ (c) $\frac{11}{16}$ (d) $\frac{1}{16}$

43. A random variable X has the probability distribution

X	1	2	3	4	5	6	7	8
$P(X)$	0.15	0.23	0.12	0.10	0.20	0.08	0.07	0.05

For the events $E = \{X$ is a prime number$\}$ and $F = \{X < 4\}$, then $P(E \cup F)$ is equal to

(a) 0.77 (b) 0.87 (c) 0.35 (d) 0.50

44. Two cards are drawn successively with replacement from a well shuffled deck of 52 cards, then the mean of the number of aces is

(a) $\frac{1}{13}$ (b) $\frac{3}{13}$ (c) $\frac{2}{13}$ (d) None

45. If X follows a binomial distribution with parameters $n = 100$ and $p = \frac{1}{3}$, then $P(X = r)$ is maximum, when r is equal to

(a) 16 (b) 32

(c) 33 (d) None of these

46. A dice is thrown 100 times, getting an even number is considered a success. The variance of the number of successes is

(a) 10 (b) 25 (c) 18 (d) 10

47. If in a trial the probability of success is twice the probability of failure. In six trials the probability of atleast four successes is

(a) $\frac{496}{729}$ (b) $\frac{500}{729}$

(c) $\frac{400}{729}$ (d) $\frac{600}{729}$

48. If m and σ^2 are the mean and variance of the random variable X, whose distribution is given by

X	0	1	2	3
$P(X)$	$\frac{1}{3}$	$\frac{1}{2}$	0	$\frac{1}{6}$

Then,

(a) $m = \sigma^2 = 2$ (b) $m = 1, \sigma^2 = 2$

(c) $m = \sigma^2 = 1$ (d) $m = 2, \sigma^2 = 1$

49. Two points are taken at random on the given straight line segment of length a. The probability for the distance between them to exceed a given length c, where $0 < c < a$, is

(a) $\left(1 - \frac{c}{a}\right)^2$ (b) $\left(1 - \frac{a}{c}\right)^2$

(c) $1 - \frac{c^2}{a^2}$ (d) $1 - \frac{a^2}{c^2}$

50. If $a \in [-20, 0]$, then probability that the equation $16x^2 + 8(a + 5)x - 7a - 5 = 0$ has imaginary roots, is

(a) $\frac{17}{20}$ (b) $\frac{13}{20}$ (c) $\frac{7}{20}$ (d) $\frac{3}{20}$

Answers

1. (c)	**2.** (b)	**3.** (c)	**4.** (d)	**5.** (d)	**6.** (d)	**7.** (d)	**8.** (b)	**9.** (a)	**10.** (a)
11. (c)	**12.** (c)	**13.** (b)	**14.** (c)	**15.** (a)	**16.** (a)	**17.** (d)	**18.** (a)	**19.** (a)	**20.** (b)
21. (c)	**22.** (c)	**23.** (c)	**24.** (c)	**25.** (d)	**26.** (d)	**27.** (c)	**28.** (d)	**29.** (c)	**30.** (c)
31. (a)	**32.** (d)	**33.** (d)	**34.** (b)	**35.** (b)	**36.** (c)	**37.** (a)	**38.** (b)	**39.** (d)	**40.** (d)
41. (a)	**42.** (c)	**43.** (a)	**44.** (c)	**45.** (c)	**46.** (b)	**47.** (a)	**48.** (c)	**49.** (a)	**50.** (b)

Formula at a Glance

1. $P(A) + P(\bar{A}) = 1,\ 0 \le P(A) \le 1$

2. If $(m + n + p)$ equally likely, mutually exclusive and exhaustive sample points are in favour of an event A. Then, $m : n + p$ is called odds in favour of E and $n + p : m$ is called odd against.

$$\therefore \quad P(E) = \frac{m}{m + n + p}$$

3. (i) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

(ii) $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C)$

(iii) If A and B are mutually exclusive events, then

$P(A \cap B) = 0$

$\therefore P(A \cup B) = P(A) + P(B)$

4. Let A and B be two events associated to a random experiment. Then,

(i) $P(\bar{A} \cap B) = P(B) - P(A \cap B)$

(ii) $P(A \cap \bar{B}) = P(A) - P(A \cap B)$

(iii) If $B \subset A$, then

$P(A \cap \bar{B}) = P(A) - P(B)$ and $P(B) < P(A)$

(iv) If $A \subset B$, then

$P(\bar{A} \cap B) = P(B) - P(A)$ and

$P(A) < P(B)$

(v) P(exactly one of A, B occurs)

$= P(A \cap \bar{B}) + P(\bar{A} \cap B)$

$= P(A) + P(B) - 2P(A \cap B)$

(vi) P(neither A nor B occurs)

$= P(\bar{A} \cap \bar{B}) = 1 - P(A \cup B)$

5. (i) P (atleast two of A, B, C occur)

$= P(A \cap B) + P(B \cap C) + P(C \cap A) - 2P(A \cap B \cap C)$

(ii) P(exactly two of A, B, C occur)

$= P(A \cap B) + P(B \cap C) + P(A \cap C) - 3P(A \cap B \cap C)$

(iii) P(exactly one of A, B, C occurs)

$= P(A) + P(B) + P(C) - 2P(A \cap B) - 2P(B \cap C) - 2P(A \cap C) + 3P(A \cap B \cap C)$

6. Probability that exactly r letters are in right envelopes

$$= \frac{1}{r!}\left[\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \ldots + (-1)^{n-r}\frac{1}{(n-r)!}\right]$$

7. (i) If $E_1, E_2, \ldots, E_n$ are independent events, then

$P(E_1 \cap E_2 \cap E_3 \cap \ldots \cap E_n) = P(E_1)\,P(E_2) \ldots P(E_n)$

(ii) If $E_1, E_2, \ldots, E_n$ are exhaustive events, then

$P(E_1 \cup E_2 \cup \ldots \cup E_n) = 1$

8. (i) $P\left(\frac{A}{B}\right) = \frac{P(A \cap B)}{P(B)},\ P(B) \ne 0$

(ii) If A and B are independent events, then

$P\left(\frac{A}{B}\right) = P(A)$

(iii) If A and B are events such that $B \ne 0$, then

$P\left(\frac{A}{B}\right) + P\left(\frac{A'}{B}\right) = 1$

9. If $E_1, E_2, \ldots, E_n$ are mutually exclusive and exhaustive events and A is any event, then

(i) $P(A) = P(E_1)P\left(\frac{A}{E_1}\right) + P(E_2)P\left(\frac{A}{E_2}\right) + \ldots + P(E_n)P\left(\frac{A}{E_n}\right)$

(ii) $P\left(\frac{E_i}{A}\right) = \frac{P(E_i)P\left(\frac{A}{E_i}\right)}{\sum_{i=1}^{n} P(E_i)P\left(\frac{A}{E_i}\right)}$ (Baye's theorem)

10. If X is a discrete random variable which assumes values $x_1, x_2, x_3, \ldots, x_n$ with respective probabilities $P_1, P_2, P_3, \ldots, P_n$. Then,

(i) Mean $(\bar{X}) = P_1x_1 + P_2x_2 + P_3x_3 + \ldots + P_nx_n = \sum_{i=1}^{n} P_ix_i$

(ii) Variance, $\text{var}(x) = P_1(x_1 - \bar{X})^2 + P_2(x_2 - \bar{X})^2 + \ldots + P_n(x_n - \bar{X})^2$

$= \sum_{i=1}^{n} P_i(x_i - \bar{X})^2$

(iii) Sum of total probability distribution is one.

11. In binomial distribution

(i) $P(X = r) = {}^nC_r p^r q^{n-r};\ r = 0, 1, 2, \ldots, n;\ p + q = 1$

(ii) Mean $= np$

(iii) Variance, $\sigma^2 = npq$

(iv) Standard deviation, $\sigma = \sqrt{(npq)}$

(v) Mean > Variance

(vi) Let x be a mode, then $np + p > x > np - q$

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Questions to Measure Your Problem Solving Skills

1. The solution of the equation $2x^{\log_4 3} + 3^{\log_2 \sqrt{x}} = 27$ is

(a) 4 *(b)* 12 *(c)* 16 *(d)* 8

2. The coefficient of middle term in the binomial expansion in power of x of $(1+\alpha x)^6$ and of $(1-\alpha x)^8$ is same, then α is equal to

(a) $\frac{-3}{7}$ *(b)* $\frac{-2}{7}$ *(c)* $\frac{2}{7}$ *(d)* $\frac{3}{7}$

3. The value of $\cos(2\cos^{-1} x + \sin^{-1} x)$ at $x = \frac{2}{3}$ is

(a) $-\frac{\sqrt{5}}{3}$ *(b)* $\frac{\sqrt{5}}{3}$

(c) $\frac{\sqrt{5}}{9}$ *(d)* $\frac{2\sqrt{5}}{3}$

4. If $y' = y + 1$ and $y(0) = 1$, then value of $y(\log 2)$ is

(a) 1 *(b)* 2 *(c)* 3 *(d)* 4

5. The range of x for which the vectors $-x^2\hat{i} + \hat{j} + \hat{k}$, $\hat{i} - x^2\hat{j} + \hat{k}$ and $\hat{i} + \hat{j} - x^2\hat{k}$ are coplanar is a subset of

(a) (2, 3) *(b)* (0, 1)

(c) $(-1, 0) \cup (1, 2)$ *(d)* $(-2, -1) \cup \left(\frac{1}{2}, \infty\right)$

6. If $I_1 = \int_0^1 \frac{e^x}{3+x}\,dx$ and $I_2 = \int_0^1 \frac{x^3}{e^{x^4}(4-x^4)}\,dx$, then $\frac{I_1}{I_2}$ is

(a) 4 *(b)* 4e *(c)* 3e *(d)* 2e

7. There are 35 seats in forum for all which three political parties are contesting, the number of ways in which no party gets absolute majority is

(a) 150 *(b)* 151

(c) 152 *(d)* 153

8. If $|z| = 1$ and $z \neq \pm 1$, then all the values of $\frac{z}{1-z^2}$ lie on

(a) the circle $|z| = 2$

(b) a line not passing through the origin

(c) the real axis

(d) the imaginary axis

9. The parabola $y^2 = 8x$ and the circle $x^2 + y^2 = 2$

(a) have only two common tangents which are at right angles

(b) have only two common tangents which are inclined at 45°

(c) have infinitely many common tangents

(d) do not have any common tangents

10. An ordinary cube has 4 blank faces, one face marked 3 and the other 4. The probability of obtaining 16 in five throws is

(a) $\frac{5}{6^5}$ *(b)* $\frac{25}{6^5}$

(c) $\frac{2}{6^5}$ *(d)* $\frac{15}{6^5}$

11. If $\sin\alpha + \cos\beta = \frac{1}{\sqrt{2}}$ and $\cos\alpha + \sin\beta = \sqrt{\frac{2}{3}}$, then the value of $\tan\frac{\alpha-\beta}{2}$ is

(a) $3\sqrt{3} - 6$ *(b)* $\sqrt{3} - 2$

(c) $4\sqrt{3} - 7$ *(d)* $4\sqrt{3} + 7$

12. If $\frac{a}{\sqrt{bc}} - 2 = \sqrt{\frac{b}{c}} + \sqrt{\frac{c}{b}}$ where $a, b, c > 0$, then family of lines $\sqrt{a}x + \sqrt{b}y + \sqrt{c} = 0$ passes through the point

(a) (1, 1) *(b)* (1, −2)

(c) (−1, 2) *(d)* (−1, 1)

13. Let $f: [0, \sqrt{3}] \to \left[0, \frac{\pi}{3} + \log_e 2\right]$ be a function defined by $f(x) = \log_e \sqrt{x^2+1} + \tan^{-1} x$, then f is

(a) one-one and onto *(b)* one-one but onto

(c) onto but not one-one *(d)* neither one-one nor onto

14. The sum of the series $\frac{7}{6}\left(\frac{1}{3}\right) + \frac{9}{12}\left(\frac{1}{3}\right)^2 + \frac{11}{20}\left(\frac{1}{3}\right)^3 + \ldots$ upto 10 terms is equal to

(a) $\frac{1}{2} - \frac{1}{12 \times 3^{10}}$ *(b)* $\frac{1}{3} - \frac{1}{12 \times 3^{10}}$

(c) $\frac{1}{2} - \frac{1}{10 \times 3^{10}}$ *(d)* $\frac{1}{2} + \frac{1}{12 \times 3^{10}}$

15. Sum of all the third order determinant that can be made with real and distinct numbers 1, 2, 3, ..., 9 is equal to
(a) 9 times value of any one of the determinant
(b) 9! times value of any one of the determinant
(c) 9^9 times the value of any one of the determinants
(d) None of the above

16. If $f : R \to R$ is a differentiable function and $f(1) = 4$. Then, the value of $\lim\limits_{x \to 1} \int_4^{f(x)} \frac{2t}{x-1}\, dt$ is
(a) $8f'(1)$ *(b)* $4f'(1)$
(c) $2f'(1)$ *(d)* $f'(1)$

17. The area of the portion common to $y = \sin^{-1}(\sin x)$ and $y = [\sin^{-1}(\sin x)]$ in $[0, \pi]$, where [.] denotes the greatest integer function, is
(a) $(\pi - 2)^2$ *(b)* $\frac{(\pi - 2)^2}{2}$
(c) $\frac{1}{4}(\pi - 2)^2$ *(d)* None of these

18. If straight line $\frac{ax}{3} + \frac{by}{4} = c$ is a normal to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 (a > b)$, then $a^2 - b^2$ is equal to
(a) 4 *c* *(b)* 5 *c*
(c) 6 *c* *(d)* None of these

19. The function f is defined in the following way $f(x) = \int_{x-1}^{x+1} \frac{dt}{1+t^8}$, which of the following option is correct?
(a) f has no extremum
(b) f has a local maximum for some $x \in (0, 1)$
(c) f has a local minimum for some $x \in (-1, 0)$
(d) f has a local maximum at $x = 0$

20. If the eccentricity of the hyperbola $x^2 - y^2 \sec^2 \alpha = 5$ is $\sqrt{3}$ times the eccentricity of the ellipse $x^2 \sec^2\alpha + y^2 = 25$, then the value of α is
(a) $\frac{\pi}{6}$ *(b)* $\frac{\pi}{4}$
(c) $\frac{\pi}{3}$ *(d)* $\frac{\pi}{2}$

21. A relation R on the set of complex numbers is defined by $z_1 R z_2 \Leftrightarrow \frac{z_1 - z_2}{z_1 + z_2}$ is real, then R is an
(a) equivalence relation *(b)* only reflexive
(c) only transitive *(d)* None of these

22. If $I_n = \int (\log x)^n\, dx$, then $I_n + nI_{n-1}$ is equal to
(a) $x(\log x)^n$ *(b)* $(x \log x)^n$
(c) $(\log x)^{n-1}$ *(d)* $n(\log x)^n$

23. The angle between the lines $2x = 3y = -z$ and $6x = -y = -4z$ is
(a) $\frac{\pi}{6}$ *(b)* $\frac{\pi}{4}$ *(c)* $\frac{\pi}{3}$ *(d)* $\frac{\pi}{2}$

24. The statement $p \to (q \to p)$ is equivalent to
(a) $p \to (p \to q)$ *(b)* $p \to (p \vee q)$
(c) $p \to (p \wedge q)$ *(d)* $p \to (p \leftrightarrow q)$

25. The number of integral value of x satisfying $\sqrt{-x^2 + 10x - 16} < x - 2$ is
(a) 0 *(b)* 1 *(c)* 2 *(d)* 3

26. Let $x_1, x_2, ..., x_n$ be n observation such that $\Sigma x_i^2 = 400$ and $\Sigma x_i = 80$. Then, a possible value of n among the following is
(a) 12 *(b)* 9
(c) 18 *(d)* 15

27. If $2z_1 - 3z_2 + z_3 = 0$, then z_1, z_2 and z_3 are represented by
(a) three vertices of triangle *(b)* three vertices of a rhombus
(c) three collinear points *(d)* None of these

28. Let $g(x) = 1 + x - [x]$, (where [ ] is greatest integer function) and $f(x) = \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0 \end{cases}$, then for all x, $f(g(x))$ is equal to
(a) x *(b)* 1
(c) $f(x)$ *(d)* $g(x)$

29. The number of solutions of the equation $\cos(\pi\sqrt{x-4}) \cos(\pi\sqrt{x}) = 1$ is
(a) 0 *(b)* 1
(c) 2 *(d)* infinite

30. Suppose families always have one, two or three children, with probabilities $\frac{1}{4}, \frac{1}{2}$ and $\frac{1}{4}$, respectively. Assume everyone eventually gets married and has children, the probability of a couple having exactly four grandchildren, is
(a) $\frac{27}{128}$ *(b)* $\frac{37}{128}$
(c) $\frac{25}{128}$ *(d)* $\frac{20}{128}$

Answers

1. (c)	2. (c)	3. (a)	4. (c)	5. (d)	6. (b)	7. (d)	8. (d)	9. (a)	10. (b)
11. (c)	12. (d)	13. (a)	14. (a)	15. (d)	16. (a)	17. (c)	18. (b)	19. (d)	20. (b)
21. (a)	22. (a)	23. (d)	24. (b)	25. (d)	26. (c)	27. (c)	28. (b)	29. (b)	30. (a)

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Mock Questions with Complete Solutions

1. The expression $\tan\left(\frac{1}{2}\log\left(\frac{a-ib}{a+ib}\right)\right)$ reduces to

(a) $\frac{ab}{a^2+b^2}$ (b) $\frac{2ab}{a^2-b^2}$

(c) $\frac{ab}{a^2-b^2}$ (d) None of these

2. Suppose $a, b > 0$ and α, β, γ $(\alpha > \beta > \gamma)$ are the roots of $\frac{x-a}{b}+\frac{x-b}{a}=\frac{b}{x-a}+\frac{a}{x-b}$ and $\alpha-\beta-\gamma=c$, then a, c, b are in

(a) AP (b) GP

(c) HP (d) None of these

3. Number of six digit numbers which are multiples of 3 and begin and end with 3 is

(a) 3332 (b) 3334 (c) 9999 (d) 3333

4. The equation of asymptote of the curve $2x^2+5xy+2y^2+4x+5y=0$ is given by

(a) $2x^2+5xy+2y^2+4x+5y+2=0$

(b) $2x^2+5xy+2y^2+4x+5y-2=0$

(c) $2x^2+5xy+2y^2+4x+5y-4=0$

(d) None of the above

5. Let $\vec{a}=2\hat{i}+\hat{j}+\hat{k}$, $\vec{b}=\hat{i}+2\hat{j}-\hat{k}$ and a unit vector $\vec{c}$ be coplanar. If $\vec{c}$ is perpendicular to $\vec{a}$, then $\vec{c}$ is equal to

(a) $\frac{-\hat{j}+\hat{k}}{\sqrt{2}}$ (b) $\frac{(-\hat{i}-\hat{j}-\hat{k})}{\sqrt{3}}$

(c) $\frac{\hat{i}-2\hat{j}}{\sqrt{5}}$ (d) $\frac{\hat{i}-\hat{j}-\hat{k}}{\sqrt{3}}$

6. In ΔABC, $2a^2+4b^2+c^2=4ab+2ac$, then the numerical value of $\cos B$ is

(a) $\frac{5}{8}$ (b) $\frac{3}{8}$

(c) $\frac{7}{8}$ (d) None of these

7. If every element of a square non-singular matrix A is multiplied by k and the new matrix is denoted by B, then $|A^{-1}|$ and $|B^{-1}|$ are related as

(a) $|A^{-1}|=k|B^{-1}|$ (b) $|A^{-1}|=\frac{1}{k}|B^{-1}|$

(c) $|A^{-1}|=k^n|B^{-1}|$ (d) $|A^{-1}|=\frac{1}{k^n}|B^{-1}|$

8. Let R be the region in the first quadrant bounded by X and Y-axes and the graph of $f(x)=\frac{9}{25}x+b$ and $y=f^{-1}(x)$. If the area of R is 49, then the value of b is

(a) $\frac{25}{9}$ (b) $\frac{28}{5}$

(c) $\frac{5}{28}$ (d) None of these

9. Let $f(x)=\sin^{23}x-\cos^{22}x$ and $g(x)=1+\frac{1}{2}\tan^{-1}|x|$, then the number of real values of x in interval $[-10\pi, 20\pi]$ satisfying the equation $f(x)=[g(x)]$, where $[.]$ denotes the greatest integer function, is

(a) 6 (b) 10

(c) 15 (d) 20

10. Let set $A=\{1, 2, 3, \ldots 22\}$. Set B is a subset of A and B has exactly 11 elements, then the sum of elements of all possible subsets B is

(a) $252\times{}^{21}C_{11}$ (b) $230\times{}^{21}C_{10}$

(c) $253\times{}^{21}C_{9}$ (d) $253\times{}^{21}C_{10}$

11. An urn contains five balls. Two balls are drawn and found to be white. Probability that all balls are white, is

(a) $\frac{1}{3}$ (b) $\frac{2}{9}$

(c) $\frac{1}{2}$ (d) $\frac{3}{4}$

12. If $y = f(x)$ is a real-valued twice differentiable function defined on R, then $\frac{d^2y}{dx^2}\left(\frac{dx}{dy}\right)^3 + \frac{d^2x}{dy^2}$ is equal to

(a) $\frac{dy}{dx}$ (b) $\frac{dx}{dy}$ (c) $\frac{d^2y}{dx^2}$ (d) 0

13. If the ellipse $\frac{x^2}{a^2-3} - \frac{y^2}{a+4} = 1$ is inscribed in a square of side length $\sqrt{2}\,a$, then a is equal to

(a) -1 (b) -2
(c) 1 (d) None of these

14. If the lines $\frac{x-1}{1} = \frac{y+3}{-k} = \frac{z-1}{k}$ and $\frac{2x}{1} = \frac{y-1}{1} = \frac{z-2}{-1}$ are coplanar, then the value of k is

(a) 1 (b) -1
(c) 2 (d) -2

15. The differential equation representing the family of curves $y^2 = 2c(x + \sqrt{c})$, where c is a positive parameter, is

(a) order 1, degree 3
(b) order 2, degree 3
(c) order 1, degree 2
(d) order 1, degree 1

16. If $\int_2^{-1} \frac{\lambda^2\, d\lambda}{\sqrt{\lambda^4 - \lambda^2 + 1}} = I$, then the value of $\int_2^{-1} \frac{x\,dx}{\sqrt{(x - x^{-1})^2 + 1}} - \int_1^{-1/2} \frac{dx}{x^3\sqrt{x^2 + x^{-2} - 1}}$ is

(a) zero (b) I
(c) $-I$ (d) $-2I$

17. The value of $\lim_{x \to 0}\left[(1 - e^x)\frac{\sin x}{|x|}\right]$; where [.] denotes the greatest integer function, is

(a) 0 (b) 1
(c) -1 (d) Does not exist

18. If $\int \frac{x^3 - 8}{x^3\sqrt{x^4 - x^3 + 2}}\,dx = \frac{(k^2 - 7k + 14)\sqrt{x^4 - x^3 + 2}}{x^2} + C$, then the values of k are

(a) 3 and 4 (b) -3 and -4
(c) 3 and -4 (d) -3 and 4

19. The acute angle of a rhombus whose side is geometric mean of its diagonals, is

(a) 15° (b) 30° (c) 20° (d) 80°

20. If $x - 2y + 4 = 0$ and $2x + y - 5 = 0$ are the sides of a isosceles triangle having area 10 sq units, then the equation of third side is

(a) $3x - y = -9$
(b) $3x - y + 11 = 0$
(c) $x - 3y = 19$
(d) $3x - y + 15 = 0$

21. The range of values of r for which the point $\left(-5 + \frac{r}{\sqrt{2}}, -3 + \frac{r}{\sqrt{2}}\right)$ is an interior point of the major segment of the circle $x^2 + y^2 = 16$, cut-off by the line $x + y = 2$ is

(a) $(-\infty, 5\sqrt{2})$
(b) $(4\sqrt{2} - \sqrt{14}, 5\sqrt{2})$
(c) $(4\sqrt{2} - \sqrt{14}, 4\sqrt{2} + \sqrt{14})$
(d) None of the above

22. If $\sin^2\theta = \frac{x^2 + y^2 + 1}{2x}$, then x must be

(a) -3 (b) -2
(c) $+1$ (d) None of these

23. The number of roots of the equation $2^{\tan\left(x - \frac{\pi}{4}\right)} - 2(0.25)^{\frac{\sin^2\left(x - \frac{\pi}{4}\right)}{\cos 2x}} + 1 = 0$ is

(a) 0 (b) 1 (c) 2 (d) 3

24. The value of $\frac{\alpha^3}{2}\operatorname{cosec}^2\left(\frac{1}{2}\tan^{-1}\frac{\alpha}{\beta}\right) + \frac{\beta^3}{2}\sec^2\left(\frac{1}{2}\tan^{-1}\frac{\beta}{\alpha}\right)$ is

(a) $(\alpha - \beta)(\alpha^2 + \beta^2)$ (b) $(\alpha + \beta)(\alpha^2 - \beta^2)$
(c) $(\alpha + \beta)(\alpha^2 + \beta^2)$ (d) None of these

25. If $f(x) = \cos^{-1}\left(\frac{2x}{1 + x^2}\right)$, then $f(x)$ is differentiable on

(a) $(-\infty, \infty)$ (b) $(-\infty, \infty) \sim \{0\}$
(c) $(-\infty, \infty) - \{-1, 1\}$ (d) None of these

26. If $x, y \in [0, 10]$, then number of solutions (x, y) of the inequation $3^{\sec^2 x - 1}\sqrt{9y^2 - 6y + 2} \le 1$ is

(a) 2 (b) 1
(c) 4 (d) infinite

27. The sum of $\sum_{r=0}^{n}(-1)^r \frac{{}^nC_r}{{}^{r+2}C_r}$ is

(a) $\frac{2}{n+1}$ (b) $\frac{1}{n+2}$ (c) $\frac{2}{n-2}$ (d) $\frac{2}{n+2}$

28. The set of values of 'h' for which the number of distinct common normals of $(x-2)^2 = 4(y-3)$ and $x^2 + y^2 - 2x - hy - c = 0\,(c > 0)$ is 3, is

(a) $(2, \infty)$ (b) $(4, \infty)$
(c) $(2, 4)$ (d) $(10, \infty)$

29. The coordinates of the point on the parabola $y^2 = 8x$ which is at a minimum distance from the circle $x^2 + (y + 6)^2 = 1$ are

(a) (2, 4) (b) (2, – 4)
(c) (–2, 4) (d) None of these

30. The statement $\sim(p \leftrightarrow \sim q)$ is

(a) equivalent to $p \leftrightarrow q$
(b) equivalent to $\sim p \leftrightarrow q$
(c) a tautology
(d) a fallacy

Answer *with* Explanations

1. *(b)* We have, $\tan\left(i\log\left(\frac{a-ib}{a+ib}\right)\right)$

Let $a+ib = re^{i\theta}$, then $\tan\theta = \frac{b}{a}$

$\therefore \quad \tan\left(i\log\left(\frac{a-ib}{a+ib}\right)\right) = \tan(i\log e^{-2i\theta})$

$= \tan(-2i^2\theta) = \tan 2\theta$

$= \frac{2\tan\theta}{1-\tan^2\theta} = \frac{\frac{2b}{a}}{1-\left(\frac{b}{a}\right)^2}$

$= \frac{2ab}{a^2-b^2}$

2. *(c)* We have,

$$\frac{x-a}{b}+\frac{x-b}{a} = \frac{b}{x-a}+\frac{a}{x-b}$$

$\Rightarrow \quad \frac{x-a}{b}-\frac{b}{x-a} = \frac{a}{x-b}-\frac{x-b}{a}$

$\Rightarrow \quad \frac{(x-a)^2-b^2}{b(x-a)} = \frac{a^2-(x-b)^2}{a(x-b)}$

$\Rightarrow \frac{(x-a-b)(x-a+b)}{b(x-a)} = \frac{(a-x+b)(a+x-b)}{a(x-b)}$

$\Rightarrow (x-a-b)(x-a+b)\,a(x-b)$

$= b(x-a)(a-x+b)(a+x-b)$

$\Rightarrow (x-a-b)\{a(x-b)(x-a+b)$

$+ b(x-a)(x+a-b)\} = 0$

$\Rightarrow x(x-a-b)\{(a+b)x-(a^2+b^2)\} = 0$

$\Rightarrow \quad x = 0,\ x = a+b \text{ or } x = \frac{a^2+b^2}{a+b}$

But $\frac{a^2+b^2}{a+b} = \frac{(a+b)^2-2ab}{a+b} = a+b-\frac{2ab}{a+b}$

$\therefore \quad \frac{a^2+b^2}{a+b} < a+b$

$\therefore \quad \alpha = a+b, \beta = \frac{a^2+b^2}{a+b}, \gamma = 0$

Given, $\alpha-\beta-\gamma = c$

$$a+b-\left(\frac{a^2+b^2}{a+b}\right) = c$$

$\Rightarrow \quad c = \frac{2ab}{a+b}$

Hence, a, c and b are in HP.

3. *(b)* : 3 *a b c d* 3

$a+b+c+d$ should be multiple of 3, $abcd$ varies from 0000 to 9999. These are in AP.

Number of values of $abcd$

$$= \frac{9999-0000}{3}+1 = 3334$$

4. *(a)* Asymptote of the hyperbola is

$$2x^2+5xy+2y^2+4x+5y+\lambda = 0$$

where, $\Delta = abc+2fgh-af^2-bg^2-ch^2 = 0 \Rightarrow \lambda = 2$

Hence, equation of asymptote is

$$2x^2+5xy+2y^2+4x+5y+2 = 0$$

5. *(a)* a, b and $\hat{c}$ are coplanar.

$\therefore \quad \hat{c} = a+\lambda b = (2\hat{i}+\hat{j}+\hat{k})+\lambda(\hat{i}+2\hat{j}-\hat{k})$

$\hat{c} = (2+\lambda)\hat{i}+(1+2\lambda)\hat{j}+(1-\lambda)\hat{k}$

Since, $\hat{c}$ is perpendicular to a.

$\therefore \quad \hat{c}\times a = 0 \Rightarrow (2+\lambda)2+(1+2\lambda)+(1-\lambda) = 0$

$\Rightarrow \quad \lambda = -2$

Now, $\hat{c} = -3\hat{j}+3\hat{k}$, but $\hat{c}$ is a unit vector

Hence, $\hat{c} = \frac{-\hat{j}+\hat{k}}{\sqrt{2}}$

6. *(c)* We have, $2a^2+4b^2+c^2 = 4ab+2ac$

$\Rightarrow \quad 2a^2+4b^2+c^2-4ab-2ac = 0$

$\Rightarrow \quad [a^2+(2b)^2-4ab]+(a^2+c^2-2ac) = 0$

$\Rightarrow \quad (a-2b)^2+(a-c)^2 = 0$

It is possible only when $a = 2b$ and $a = c$

$\therefore \quad a = 2b = c$

$\Rightarrow \quad \frac{a}{2} = \frac{b}{1} = \frac{c}{2} = K \Rightarrow a = 2K, b = K, c = 2K$

Now, using cosine rule, we get

$$\cos B = \frac{a^2+c^2-b^2}{2ac} = \frac{4K^2+4K^2-K^2}{2(2K)(2K)} = \frac{7}{8}$$

7. *(c)* We have, $B = kA$

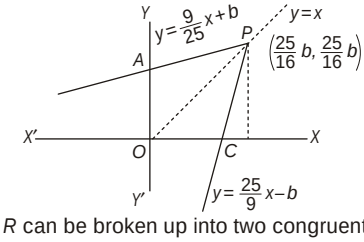
$|B| = |kA| = k^n|A|$

$\frac{1}{|B^{-1}|} = \frac{k^n}{|A^{-1}|} \Rightarrow |A^{-1}| = k^n|B^{-1}|$

8. *(b)* If $f(x) = \frac{9}{25}x+b$, then $f^{-1}(x) = \frac{25}{9}x-b$

Since, $f(x)$ and $f^{-1}(x)$ intersect at $y = x$

$\therefore$ Point of intersection is $\left(\frac{25}{16}b, \frac{25}{16}b\right)$



Region R can be broken up into two congruent ΔOAP and ΔOCP which have same base, i.e. b and same height, i.e. $\frac{25}{16}b$

$\therefore$ Area of $R = 2\times\frac{b}{2}\times\frac{25}{16}b = 49$

$\Rightarrow \quad b^2 = \frac{49\times 16}{25} \Rightarrow b = \frac{28}{5}$

9. *(c)* We have,

$$f(x) = \sin^{23} x - \cos^{22} x$$

$\Rightarrow$ $f(x) \le 1$

and $g(x) = 1 + \frac{1}{2}\tan^{-1}|x|$

$\Rightarrow$ $g(x) \ge 1 \Rightarrow [g(x)] = 1, 2, 3, 4, \ldots$

$f(x) = [g(x)]$ holds only when $f(x) = 1$ and $[g(x)] = 1$

$\therefore \left[1 + \frac{1}{2}\tan^{-1}|x|\right] = 1$

$\Rightarrow \left[\frac{1}{2}\tan^{-1}|x|\right] = 0$

$\therefore -\tan 2 < x < \tan 2$

$x = \frac{\pi}{2}$ if $x \in [0, 2\pi]$

Number of real values of x in $[-10\pi, 20\pi]$ satisfying the given equation is 15.

10. *(d)* Given, $A = \{1, 2, 3, 4, ..., 22\}$

B is a subset of A and B will have exactly 11 elements.
Number of subsets containing $1 = {}^{21}C_{10}$

$\therefore$ Sum of elements of all possible subsets of B is

$${}^{21}C_{10} \times 1 + {}^{21}C_{10} \times 2 + {}^{21}C_{10} \times 3 + \ldots + {}^{21}C_{10} \times 22$$
$$= {}^{21}C_{10}(1 + 2 + 3 + 4 + \ldots + 22)$$
$$= {}^{21}C_{10} \times \frac{22 \times 23}{2} = {}^{21}C_{10} \times 253$$

11. *(c)* Event E_1 = urn contains 5 white balls
Event E_2 = urn contains 4 white balls
Event E_3 = urn contains 3 white balls
Event E_4 = urn contains 2 white balls
Event A = drawing two white balls

$$P(E_1) = P(E_2) = P(E_3) = P(E_4) = \frac{1}{4}$$

$$P\left(\frac{A}{E_1}\right) = 1,\ P\left(\frac{A}{E_2}\right) = \frac{{}^4C_2}{{}^5C_2},\ P\left(\frac{A}{E_3}\right) = \frac{{}^3C_2}{{}^5C_2},\ P\left(\frac{A}{E_4}\right) = \frac{{}^2C_2}{{}^5C_2}$$

Required probability

$$P\left(\frac{E_1}{A}\right) = \frac{P(E_1) \times P\left(\frac{A}{E_1}\right)}{P(E_1) \times P\left(\frac{A}{E_1}\right) + P(E_2) \times P\left(\frac{A}{E_2}\right) + P(E_3) \times P\left(\frac{A}{E_3}\right) + P(E_4) \times P\left(\frac{A}{E_4}\right)}$$

$$= \frac{1 \times \frac{1}{4}}{\frac{1}{4}\left[1 + \frac{3}{5} + \frac{3}{10} + \frac{1}{10}\right]} = \frac{1}{2}$$

12. *(d)* We know that, $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$

$$= \frac{d}{dy}\left(\frac{dx}{dy}\right)^{-1} \times \frac{dy}{dx}$$
$$= -\frac{d^2x}{dy^2}\left(\frac{dx}{dy}\right)^{-2} \times \frac{dy}{dx}$$
$$= -\frac{d^2x}{dy^2} \times \left(\frac{dx}{dy}\right)^{-3}$$

$\Rightarrow$ $\frac{d^2y}{dx^2}\left(\frac{dx}{dy}\right)^3 + \frac{d^2x}{dy^2} = 0$

13. *(d)* Sides of the square will be perpendicular to the tangents of the ellipse, so vertices of square lie on director circle. So, diameter of director circle is

$$2\sqrt{(a^2 - 3) + (a + 4)} = \sqrt{2a^2 + 2a^2}$$

$\Rightarrow$ $2\sqrt{a^2 + a + 1} = 2a$

$\Rightarrow$ $a = -1$ but for ellipse to exist $a^2 > 3$ and $a > -4$, so a cannot take value (-1).

14. *(d)* Given lines are $\frac{x-1}{1} = \frac{y+3}{-k} = \frac{z-1}{k}$

and $\frac{2x}{1} = \frac{y-1}{1} = \frac{z-2}{-1}$

These lines are coplanar.

$$\therefore \begin{vmatrix} 0-1 & 1+3 & 2-1 \\ 1 & -k & k \\ 1 & 2 & -2 \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} -1 & 4 & 1 \\ 1 & -k & k \\ 1 & 2 & -2 \end{vmatrix} = 0$$

$\Rightarrow$ $-1(2k - 2k) - 4(-2 - k) + 1(2 + k) = 0$

$\Rightarrow$ $k = -2$

15. *(a)* Given equation is $y^2 = 2c(x + \sqrt{c})$ …(i)

This equation has only one arbitrary constant. Hence, order of the differential equation is 1.

On differentiating, we get

$$2yy' = 2c \Rightarrow c = yy'$$

Substituting the value of c in Eq. (i), we get

$$y^2 = 2yy'(x + \sqrt{yy'})$$
$$y = 2xy' + 2(y)^{1/2}(y')^{3/2}$$

$\Rightarrow$ $(y - 2xy')^2 = 4yy'^3$

Hence, the degree of differential equation is 3.

16. *(d)* We have, $I = \int_2^{-1} \frac{x^2 dx}{\sqrt{x^4 - x^2 + 1}}$

Let $I_1 = \int_2^{-1} \frac{x dx}{\sqrt{1 + x^2 + \frac{1}{x^2} - 2}} = \int_2^{-1} \frac{x|x|\,dx}{\sqrt{x^4 + 1 - x^2}}$

$$= -\int_2^{-1} \frac{x^2 dx}{\sqrt{x^4 + 1 - x^2}} = -I$$

and $I_2 = \int_1^{-1/2} \frac{dx}{x^3\sqrt{x^2 + \frac{1}{x^2} - 1}}$

$$= \int_{1/2}^{-1} \frac{-dx}{x^3\sqrt{x^2 + \frac{1}{x^2} - 1}}$$

Put $x = \frac{1}{t} \Rightarrow dx = -\frac{1}{t^2}dt$ $= \int_2^{-1} \frac{\frac{1}{t^2}dt}{\frac{1}{t^3}\sqrt{\frac{1}{t^2} + t^2 - 1}}$

$$= \int_{2}^{-1} \frac{t|t|}{\sqrt{t^4 - t^2 + 1}}\, dt$$

$$= \int_{2}^{-1} \frac{t^2\, dt}{\sqrt{t^4 - t^2 + 1}} = I$$

$\therefore \quad I_1 - I_2 = -I - I = -2I$

17. *(c)* We have,

$$f(x) = \begin{cases} (1 - e^x)\dfrac{\sin x}{x}; & x > 0 \\ (1 - e^x)\dfrac{\sin x}{-x}; & x < 0 \end{cases}$$

Clearly, $f(x) < 0$ for $x > 0$

$f(x) < 0$ for $x < 0$

$\Rightarrow \quad \lim\limits_{x \to 0} [f(x)] = -1$

18. *(a)* We have, $\int \frac{x^3 - 8}{x^3\sqrt{x^4 - x^3 + 2}}\, dx$

$$= \frac{(k^2 - 7k + 14)\sqrt{x^4 - x^3 + 2}}{x^2} + C$$

Taking x^4 common from the D^r, we get

$$\int \frac{x^3 - 8}{x^3 \times x^2\sqrt{1 - \frac{1}{x} + \frac{2}{x^4}}}\, dx = \int \frac{\frac{1}{x^2} - \frac{8}{x^5}}{\sqrt{1 - \frac{1}{x} + \frac{2}{x^4}}}\, dx$$

Put $\quad 1 - \frac{1}{x} + \frac{2}{x^4} = t^2$

$\Rightarrow \quad \left(\frac{1}{x^2} - \frac{8}{x^5}\right) dx = 2t\, dt$

$$\int \frac{2t\, dt}{t} = 2\int dt = 2t = 2\sqrt{1 - \frac{1}{x} + \frac{2}{x^4}}$$

$$= \frac{2\sqrt{x^4 - x^3 + 2}}{x^2}$$

On comparing, we get

$k^2 - 7k + 14 = 2$

$\Rightarrow \quad k^2 - 7k + 12 = 0$

$\Rightarrow \quad (k - 4)(k - 3) = 0$

$\therefore \quad k = 3$ and 4

19. *(b)* If the acute angle is θ, then

$$\cos(\pi - \theta) = \frac{a^2 + a^2 - d^2}{2a^2}$$

$\Rightarrow \quad 2a^2 - d^2 = -2a^2\cos\theta$

$\Rightarrow \quad d_1^2 = 4a^2\cos^2\frac{\theta}{2}$

$\Rightarrow \quad d_1 = 2a\cos\frac{\theta}{2}$

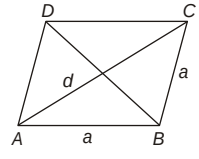
Similarly, $\quad d_2^2 = 2a^2(1 - \cos\theta)$

$\Rightarrow \quad d_2 = 2a\sin\frac{\theta}{2}$

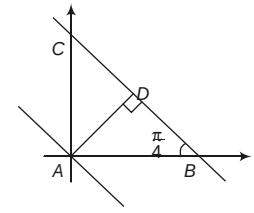
Now, we have $\quad a^2 = d_1 d_2$

$\Rightarrow \quad a^2 = 4a^2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$

$\Rightarrow \quad \sin\theta = \frac{1}{2} \Rightarrow \theta = 30°$



20. *(a)* Since, given lines are mutually perpendicular and intersect at $A\left(\frac{6}{5}, \frac{13}{5}\right)$



Equations of angle bisectors of the given lines are

$$x - 2y + 4 = \pm(2x + y - 5)$$

i.e. $\quad x + 3y = 9$ and $3x - y = 1$

Side BC will be parallel to these bisectors.

Let $\quad AD = a$

$\Rightarrow \quad AB = a\sqrt{2}$ and

Area of $\quad \Delta ABC = \frac{1}{2}(a\sqrt{2})^2 \Rightarrow a^2 = 10$

$\Rightarrow \quad a = \sqrt{10}$

Let equation of side BC be $x + 3y = \lambda$, then

$$\pm\sqrt{10} = \frac{\frac{6}{5} + \frac{39}{5} - \lambda}{\sqrt{10}}$$

$\Rightarrow \quad \lambda = -1$ and 19

Therefore, equation of BC is $x + 3y = -1$ or $x + 3y = 19$

If equation of BC is $3x - y = \lambda$, then

$$\pm\sqrt{10} = \frac{\frac{18}{5} - \frac{13}{5} - \lambda}{\sqrt{10}} \Rightarrow \lambda = -9, 11$$

Hence, equation of BC is $3x - y = -9$

or $\quad 3x - y = 11$

21. *(b)* The given point is an interior point

$\Rightarrow \quad \left(5 + \frac{r}{\sqrt{2}}\right)^2 + \left(3 + \frac{r}{\sqrt{2}}\right)^2 - 16 < 0$

$\Rightarrow \quad r^2 - 8\sqrt{2}r + 18 < 0$

$4\sqrt{2} - \sqrt{14} < r < 4\sqrt{2} + \sqrt{14}$ …(i)

The point is on the major segment.

$\Rightarrow$ The centre and the point are on the same side of the line

$x + y = 2.$

$\Rightarrow \quad -5 + \frac{r}{\sqrt{2}} - 3 + \frac{r}{\sqrt{2}} - 2 < 0$

$\Rightarrow \quad r < 5\sqrt{2}$ ...(ii)

From Eqs. (i) and (ii),

$$4\sqrt{2} - \sqrt{14} < r < 5\sqrt{2}$$

22. *(c)* Since, $\sin^2\theta \le 1$

$$\Rightarrow \quad \frac{x^2 + y^2 + 1}{2x} \le 1$$

$\Rightarrow \quad x^2 + y^2 + 1 \le 2x$

$\Rightarrow \quad (x - 1)^2 + y^2 \le 0$

It is possible, iff $x - 1 = 0$

and $\quad y = 0$

$\Rightarrow \quad x = 1$ and $y = 0$

23. *(a)* Take $\dfrac{\sin^2\left(x-\frac{\pi}{4}\right)}{\cos 2x}=\dfrac{\frac{1}{2}(\sin x-\cos x)^2}{\cos^2 x-\sin^2 x}$

$$=\frac{-\frac{1}{2}(\sin x-\cos x)}{\cos x+\sin x}$$

$$=-\frac{1}{2}\tan\left(x-\frac{\pi}{4}\right)$$

The given equation reduces to

$$2^{\tan\left(x-\frac{\pi}{4}\right)}-2\,(0.25)^{-\frac{1}{2}\tan\left(x-\frac{\pi}{4}\right)}+1=0$$

$\Rightarrow \quad 2^{\tan\left(x-\frac{\pi}{4}\right)}=1$

$\because \quad \tan\left(x-\frac{\pi}{4}\right)=0$

$\Rightarrow \quad x-\frac{\pi}{4}=0$

$\Rightarrow \quad x=\frac{\pi}{4}$

$\Rightarrow$ $x=\dfrac{\pi}{4}$ which is not possible as $\cos 2x=0$ for this value of x, hence the number of roots of the given equation is zero. Original equation is not defined.

24. *(c)* We have,

$$\frac{\alpha^3}{2}\operatorname{cosec}^2\left(\frac{1}{2}\tan^{-1}\frac{\alpha}{\beta}\right)+\frac{\beta^3}{2}\sec^2\left(\frac{1}{2}\tan^{-1}\frac{\beta}{\alpha}\right)$$

$$=\alpha^3\times\frac{1}{1-\cos\left(\tan^{-1}\frac{\alpha}{\beta}\right)}+\beta^3\times\frac{1}{1+\cos\left(\tan^{-1}\frac{\beta}{\alpha}\right)}$$

$$\left[\because \frac{1}{2}\operatorname{cosec}^2\frac{\theta}{2}=\frac{1}{1-\cos\theta}\text{ and }\frac{1}{2}\sec^2\frac{\theta}{2}=\frac{1}{1+\cos\theta}\right]$$

$$=\frac{\alpha^3}{1-\cos\left[\cos^{-1}\left(\frac{\beta}{\sqrt{\alpha^2+\beta^2}}\right)\right]}+\frac{\beta^3}{1+\cos\left[\cos^{-1}\left(\frac{\alpha}{\sqrt{\alpha^2+\beta^2}}\right)\right]}$$

$$=\frac{\alpha^3}{1-\frac{\beta}{\sqrt{\alpha^2+\beta^2}}}+\frac{\beta^3}{1+\frac{\alpha}{\sqrt{\alpha^2+\beta^2}}}$$

$$=\sqrt{\alpha^2+\beta^2}\left(\frac{\alpha^3}{\sqrt{\alpha^2+\beta^2}-\beta}+\frac{\beta^3}{\sqrt{\alpha^2+\beta^2}+\alpha}\right)$$

$$=\sqrt{\alpha^2+\beta^2}\left[\frac{\alpha^3(\sqrt{\alpha^2+\beta^2}+\beta)}{\alpha^2}+\frac{\beta^3(\sqrt{\alpha^2+\beta^2}-\alpha)}{\beta^2}\right]$$

$$=\sqrt{\alpha^2+\beta^2}\,[(\alpha+\beta)\sqrt{\alpha^2+\beta^2}]$$

$$=(\alpha+\beta)(\alpha^2+\beta^2)$$

25. *(c)* We have,

$$f(x)=\cos^{-1}\left(\frac{2x}{1+x^2}\right)$$

$$f'(x)=\frac{-1}{\sqrt{1-\left(\frac{2x}{1+x^2}\right)^2}}\times\frac{d}{dx}\left(\frac{2x}{1+x^2}\right)$$

$$=\frac{-(1+x^2)}{\sqrt{(1+x^2)^2-4x^2}}\times\frac{2(1-x^2)}{(1+x^2)^2}$$

$$=\frac{-2}{1+x^2}\times\frac{1-x^2}{|1-x^2|}=\begin{cases}-\dfrac{2}{1+x^2}, & \text{if } |x|<1\\ \dfrac{2}{1+x^2}, & \text{if } |x|>1\end{cases}$$

Clearly, $f(x)$ is differentiable everywhere except at the points, where $|x|=1$, i.e. $x=\pm 1$ Hence, $f(x)$ is differentiable on $(-\infty,\infty)-\{-1,1\}$

26. *(c)* Here, $3^{\sec^2 x-1}\sqrt{9y^2-6y+2}\le 1$ can be rewritten as

$$3^{\sec^2 x}\sqrt{y^2-\frac{2y}{3}+\frac{2}{9}}\le 1$$

$\Rightarrow \quad 3^{\sec^2 x}\sqrt{\left(y-\frac{1}{3}\right)^2+\frac{1}{9}}\ge 1$

where, $\sec^2 x\ge 1$

$3^{\sec^2 x}\ge 3$

and $\sqrt{\left(y-\frac{1}{3}\right)^2+\frac{1}{9}}\ge\frac{1}{3}$

So, we should have $\sec^2 x=1$

and $y-\frac{1}{3}=0$

$\Rightarrow$ $x=0,\pi,2\pi,3\pi$ and $y=\dfrac{1}{3}$

$\therefore$ Total solutions are 4.

27. *(d)* We have,

$$\sum_{r=0}^{n}(-1)^r\frac{{}^nC_r}{{}^{r+2}C_r}$$

$$=\sum_{r=0}^{n}(-1)^r\frac{n!}{(n-r)!\,r!}\times\frac{r!\,2!}{(r+2)!}$$

$$=2\sum_{r=0}^{n}(-1)^r\frac{n!}{(n-r)!\,(r+2)!}$$

$$=\frac{2}{(n+1)(n+2)}\sum_{r=0}^{n}(-1)^r\frac{(n+2)!}{\{(n+2)-(r+2)\}!\,(r+2)!}$$

$$=\frac{2}{(n+1)(n+2)}\sum_{r=0}^{n}(-1)^{r+2}\,{}^{n+2}C_{r+2}$$

Put $r+2=s$

$$=\frac{2}{(n+1)(n+2)}\sum_{s=2}^{n+2}(-1)^s\,{}^{n+2}C_s$$

$$=\frac{2}{(n+1)(n+2)}\left[\sum_{s=0}^{n+2}(-1)^s\,{}^{n+2}C_s-({}^{n+2}C_0-{}^{n+2}C_1)\right]$$

$$=\frac{2}{(n+1)(n+2)}\times[0-(1-\{n+2\})]$$

$$=\frac{2}{(n+1)(n+2)}\times(n+1)=\frac{2}{n+2}$$

28. *(d)* The equation of any normal

$(x-2)^2 = 4(y-3)$ is

$x - 2 = m(y-3) - 2m - m^3$

If it passes through $\left(1, \frac{h}{2}\right)$ then

$1 - 2 = m\left(\frac{h}{2} - 3\right) - 2m - m^3$

$\Rightarrow \quad 2m^3 + m(10-h) - 2 = 0$

This equation will give three distinct values of m, if

$f'(m) = 0$ has two distinct roots,

where $f(m) = 2m^3 + m(10-h) - 2$

Now, $f'(m) = 6m^2 + 10 - h$

$f'(m) = 0$

$\Rightarrow \quad m = \pm\sqrt{\frac{h-10}{6}}$

The values of m are real and distinct,

if $h > 10$, i.e. $h \in (10, \infty)$

29. *(b)* Let $P(2t^2, 4t)$ be any point on the parabola.

Then, $CP^2 = (2t^2)^2 + (4t+6)^2$

$= 4t^4 + 4(2t+3)^2$

Let $f(t) = 4t^4 + 4(2t+3)^2$

$f'(t) = 16t^3 + 16(2t+3)$

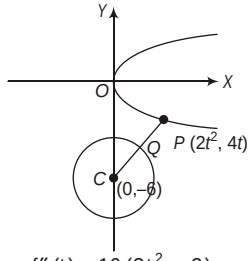
$= 16(t^3 + 2t + 3)$

$= 16(t+1)(t^2 - t + 3)$

$= 16(t+1)\left[\left(t - \frac{1}{2}\right)^2 + \frac{11}{4}\right]$

$\therefore \quad f'(t) = 0$

$\Rightarrow \quad t = -1$



Also, $f''(t) = 16(3t^2 + 2)$

$\Rightarrow \quad f''(-1) = 16(3+2) = 80 > 0$

$\because f(t)$ is least when $t = -1$

Thus, required coordinates are $(2, -4)$.

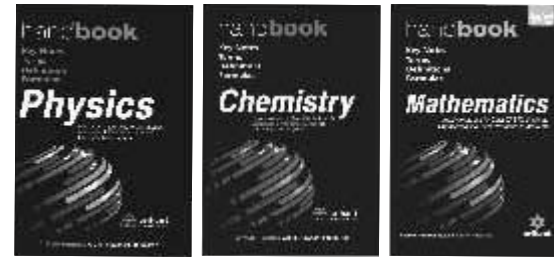
30. *(a)*

p	q	$\sim p$	$\sim q$	$p \leftrightarrow q$	$p \leftrightarrow \sim q$	$\sim p \leftrightarrow q$	$\sim (p \leftrightarrow \sim q)$
T	F	F	T	F	T	T	F
F	T	T	F	F	T	T	F
T	T	F	F	T	F	F	T
F	F	T	T	T	F	F	T

$\therefore \quad \sim (p \leftrightarrow \sim q)$ is equivalent to $(p \leftrightarrow q)$.

handbook

- Key Notes
- Terms
- Definitions
- Formulae



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Questions to Measure Your Problem Solving Skills

Paper I

Single Integer Answer Type

1. If the planes $x - cy - bz = 0$, $cx - y + az = 0$ and $bx + ay - z = 0$ pass through a line, then find the value of $a^2 + b^2 + c^2 + 2abc$.

2. If $f^2(x) \cdot f\left(\frac{1-x}{1+x}\right) = x^3$, $(x \neq \pm 1$ and $f(x) \neq 0)$, then the value of $|[f(-2)]|$ is (where, $[\cdot]$ is the greatest integer function)

3. If $\int_0^{100} f(x)dx = 7$, then $\sum_{r=1}^{100} \left(\int_0^1 f(r-1+x)\,dx\right)$ is equal to

4. If the largest term of the sequence $\frac{1}{503}, \frac{4}{524}, \frac{9}{581}, \ldots$ is t_p. Then, the value of p is

5. The possible number of ordered triads (m, n, p), where $m, n, p \in N$ is $(6250\,k)$, such that $1 \le m \le 100, 1 \le n \le 50, 1 \le p \le 25$ and $2^m + 2^n + 2^p$ is divisible by 3, then k is

6. All the values of parameter m for which the function $\frac{x^3}{3} + (m-1)x^2 + (m+5)x + 7 = 0$ is increasing for all $x > 0$ lie in the interval $[-k, \infty)$, then the numerical value of k is

7. Let $f(x) = x + \cos x + 2$ and $g(x)$ be the inverse function of $f(x)$, then $g'(3)$ equals

8. Let z be a complex number satisfying $|2z + 3i| = |z^2|$, if M and m are the maximum and minimum value of z, then the value of $|M + m|$ is

One or More than One Options Correct Type

9. If $f(x) = \frac{\sin \pi [x]}{\{x\}}$, [where $[x]$ and $\{x\}$ are greatest integer and fractional part of x respectively], then $f(x)$ is
(a) periodic with fundamental period 2
(b) even (c) range is singleton (d) odd

10. If p is the length of median from the vertex A to side BC of ΔABC, then
(a) $4p^2 = 2b^2 + 2c^2 - a^2$
(b) $4p^2 = b^2 + c^2 + 2bc \cos A$
(c) $4p^2 = a^2 + 4bc \cos A$
(d) $4p^2 = (2s - a)^2 - 4bc \sin^2 \frac{A}{2}$

11. Two whole numbers are randomly selected and multiplied. Consider two events E_1 and E_2 which are defined as E_1 = Their product is divisible by 5
E_2 = Units place in their product is 5
Which of the following statement is/are correct?
(a) E_1 is twice as likely to occur as E_2
(b) E_1 and E_2 are disjoint
(c) $P(E_2/E_1) = \frac{1}{4}$ (d) $P(E_1/E_2) = 1$

12. Let e be the eccentricity of a hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and $f(e)$ be the eccentricity of conjugate hyperbola $\frac{-x^2}{a^2} + \frac{y^2}{b^2} = 1$, then $\int_1^3 \underbrace{fff\ldots f(e)}_{10 \text{ times}}\,de$ is equal to
(a) 2, if n is even (b) 4, if n is even
(c) $2\sqrt{2}$, if n is odd (d) $4\sqrt{2}$, if n is odd

13. If $a, b, c \in R - \{1\}$ and the vectors $a\hat{i} + \hat{j} + \hat{k}, \hat{i} + b\hat{j} + \hat{k}$ and $\hat{i} + \hat{j} + c\hat{k}$ are coplanar, then which of the following options are correct?
(a) $\frac{1+a}{1-a} + \frac{1+b}{1-b} + \frac{1+c}{1-c} = -1$ (b) $\frac{1+a}{1-a} + \frac{1+b}{1-b} + \frac{1+c}{1-c} = 1$
(c) $\frac{a}{1-a} + \frac{1}{1-b} + \frac{1}{1-c} = 0$ (d) $\frac{1}{1-a} + \frac{1}{1-b} + \frac{1}{1-c} = 1$

14. The number of subsets of the sets $\{1, 2, 3, \ldots, n\}$ having least element i and greatest element j, $1 \le i < j < n$ is equal to
(a) 2^{25-j-i}, if $n = 25$ (b) 2^{j-i-1}, if $n = 30$
(c) 2^{j-i-1}, if $n = 37$ (d) 2^{30-j-i}, if $n = 30$

15. If $A(2\hat{i}+3\hat{j}+5\hat{k})$, $B(-\hat{i}+3\hat{j}+2\hat{k})$ and $C(x\hat{i}+5\hat{j}+y\hat{k})$ are vertices of a triangle and its median through A is equally inclined to the positive direction of the axes, then

(a) x is a prime number (b) $2x - y = 4$

(c) x is an even number (d) $(x + y)$ is a prime number

16. If k_1, k_2 $(k_1 < k_2)$ are the points of discontinuity of the function $f[f\{f(x)\}]$, where $f(x) = \frac{1}{1-x}$, then a value of k for which the points (k_1, k_2) and (k, k^2) lie on the same side of $x + 2y - 3 = 0$ is

(a) -4 (b) $\frac{1}{2}$ (c) -1 (d) 2

17. If the line $ax + by + c = 0$ is a normal to the curve $xy = 1$, then

(a) $a > 0, b > 0$ (b) $a > 0, b < 0$

(c) $a < 0, b > 0$ (d) $a < 0, b < 0$

18. If M is a skew-symmetric matrix of order $n \times n$, then

(a) $\det(M - I) = \det(M + I)$

(b) $\det(M - I) = -\det(M + I)$, if n is odd

(c) $\det(M - I) = \det(M + I)$, if n is even

(d) $\det[(I - M)(I + M)^{-1}] = 1, \det(I + M) \neq 0$

Match the Column Type

19. Match the statements of Column I with the values of Column II.

	Column I	Column II
A.	The distance of the point of intersection of the planes $\mathbf{r}\cdot(2\hat{i}+\hat{j})=0, \mathbf{r}\cdot(4\hat{j}+3\hat{k})=24$ and $\mathbf{r}\cdot(12\hat{i}-7\hat{k})=24$ from the origin is	p. 256
B.	The value of λ such that the planes $x-2y-z=10, 3x+3y+\sqrt{\lambda}z=8$ and $2x-y+2z=k$ are not able to make faces of a tetrahedron is	q. 36
C.	The product of distances of any vertex from the remaining vertices of a nine-sided regular polygon inscribed in a circle of radius $\sqrt{2}$ units is	r. 81
D.	If on the complex plane the number $72+56\sqrt{3}i$, $a+b\sqrt{3}\,i$ $(a, b \in R, b > 0)$ and 0 represent vertices of an equilateral triangle, then the value of $(a+b)^2$ is	s. 144

20. Match the statements of Column I with the values of Column II.

	Column I	Column II
A.	$\int_0^1 \sin^{-1}\left[x^2+\frac{1}{2}\right]+\cos^{-1}\left[x^2-\frac{1}{2}\right]dx$ is (where, [.] denoted greatest integer function)	p. $\frac{\pi}{4}$
B.	$\int_{-1}^{1} \frac{dx}{(1+x^2)(1+e^{\tan x})}$ equals	q. $\frac{\pi}{6}$
C.	In a ΔABC, $3\sin A + 4\cos B = 6$ and $4\sin B + 3\cos A = 1$, then $\angle C$ equals	r. π
D.	The angle between the line joining the origin to the point of intersection of the line $4x + 3y = 24$ with circle $(x-3)^2 + (y-4)^2 = 25$ is	s. $\frac{\pi}{2}$

Paper II

Single Integer Answer Type

1. If $f(x)$ is an odd periodic function with period 2, then $f(4)$ is

2. If α is a positive root of $x^{11} - x^{10} + x^8 - x^7 + x^5 - x^4 + x^2 - x - 20 = 0$, then the sum of the digits in the minimum value of α^{12} is

3. A chess board has $n \times n$ squares. Two squares are chosen at random. If the probability that two squares have a common side is $\frac{2}{45}$, then n is equal to

4. If the area bounded by circle $x^2 + y^2 = 4$, the parabola $y = x^2 + x + 1$ and the curve $y = \left[\sin^2\frac{x}{4} + \cos\frac{x}{4}\right]$, (where [.] denotes the greatest integer function) and X-axis is $\left(\sqrt{3} + \frac{2\pi}{3} - \frac{1}{k}\right)$, then the numerical quantity k should be

5. Number of solutions of $x \in [0, \pi]$ satisfying the equation $\log_{\sqrt{3}} \tan x \left(\sqrt{\log_{\sqrt{3}} 3\sqrt{3} + \log_{\tan x} 3}\right) + 1 = 0$ is/are

6. Let P, Q, R, S be four points in space satisfying $|\mathbf{PQ}| = 3$, $|\mathbf{QR}| = 7$, $|\mathbf{RS}| = 11$, $|\mathbf{SP}| = 9$, then the value of $\frac{\mathbf{PS}\cdot\mathbf{QR}}{3}$ is

7. Tangents are drawn from a point P to a circle with radius r. The points of contact are A and B and $PA = x$ if $\frac{1}{x^2} + \frac{1}{r^2} = \frac{1}{9}$, then the length of chord AB is

8. Let N be a natural number if its first digit (from the left) is deleted, it gets reduced to $\frac{N}{57}$. The sum of all the digits of N is $5K$, then K is equal to

One or More than One Options Correct Type

9. Consider the integral $I_1 = \int_1^e (1+x)(x + \log x)^{100}\,dx$,

$I_2 = \int_{\sin^{-1}(1/e)}^{\pi/2} (1 + e\sin x + \log\sin x)^{101}\cos x\,dx$.

If $I_1 + \frac{eI_2}{101}$ is $\frac{e(1+e)^{101} - k}{101}$, then k is greater than or equal to

(a) 0 (b) 1 (c) 2 (d) -1

10. There is a point 'P' inside an equilateral triangle ABC of side 'x' whose distance from the vertices A, B, C are respectively 3, 4, 5. Rotate triangle and 'P' through 60° about 'C' in clockwise direction. Let A goes to A' and P to P', then

(a) Area of $\Delta PAP'$ is 6 sq units

(b) $12(\sin\angle APB) = 6$

(c) $x = \sqrt{25 + 12\sqrt{3}}$

(d) $PP' = 5$

11. If $(\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{d})\cdot(\mathbf{a}\times\mathbf{d})=0$, then which of the following may be true?
(a) **a, b, c** and **d** are necessarily coplanar
(b) **a** lies in the plane of **c** and **d**
(c) **b** lies in the plane of **c** and **d**
(d) **c** lies in the plane of **a** and **d**

12. If $1+[\sin^{-1}x]>[\cos^{-1}x]$, then (where, [.] denotes greatest integer function)
(a) $x\in(\cos 1,\sin 1)$ (b) $x\in[\sin 1, 1]$
(c) $x\in(\cos 2, 0)$ (d) $x\in(\cos 1, 1)$

13. Let $f(x)$ be the fourth degree polynomial such that $f'(0)=-6$, $f(0)=2$ and $\lim\limits_{x\to 1}\dfrac{f(x)}{(x-1)^2}=1$, then
(a) $f''(1)=2$ (b) $f'(1)=0$
(c) length of subtangent of the curve $y=f(x)$ at a point, where it cuts the Y-axis is $\dfrac{1}{3}$
(d) area bounded by the curve $y=f(x)$ and coordinated axes is $\dfrac{8}{15}$

14. The family of lines $(2\cos t+3\sin t)x+(3\cos t-5\sin t)y-(5\cos t-2\sin t)=0$ and the family of lines $(\lambda+2\mu)x+(\lambda-3\mu)y+(\mu-\lambda)=0$, where t, λ, μ are parameters and concurrent at $L(x_1, y_1)$ and $M(x_2, y_2)$ respectively. Then,
(a) $\dfrac{x_1}{y_1}+\dfrac{x_2}{y_2}=\dfrac{1}{y_2}$ (b) slope of LM is $\dfrac{x_2}{y_2}$
(c) $6x_2+y_2=2x_1+y_1$ (d) $x_1+x_2=3(y_1+y_2)$

15. Let z_1 and z_2 be two distinct complex numbers and let $z=(1-t)z_1+tz_2$ for some real number t with $0<t<1$. If arg (ω) denotes the principal argument of a non-zero complex number ω, then
(a) $|z-z_1|+|z-z_2|=|z_1-z_2|$
(b) $\arg(z-z_1)=\arg(z-z_2)$
(c) $\begin{vmatrix} z-z_1 & \bar{z}-\bar{z}_1 \\ z_2-z_1 & \bar{z}_2-\bar{z}_1 \end{vmatrix}=0$
(d) $\arg(z-z_1)=\arg(z_2-z_1)$

16. Given a natural number n, consider the function $f(x)=\dfrac{1}{(1-x)^n}-\dfrac{1}{(1+x)^n}-2nx$, then for all $0<x<1$, which of the following is/are correct
(a) $f''(x)>0$ (b) $f'(x)>0$
(c) $f(x)>0$ (d) None of these

Linked Comprehension Type

Passage I

Directions (Q. Nos. 17-18) If $f(x)=\dfrac{x+a}{bx^2+cx+2}$, where $a, b, c\in R$, $f(-1)=0$ and $y=1$ is an asymptote of $y=f(x)$, $y=f^{-1}(x)$ is inverse of $f(x)$.

17. The correct statement(s) is/are
(a) $a=1, b=1, c=0$ (b) $a=1, b=0, c=1$
(c) $f(x)=\dfrac{x+1}{x+2}$ (d) $f(x)=\dfrac{x+1}{x^2+2}$

18. If $f^{-1}: R-\{1\}\to R$, then the correct statements is/are
(a) $f^{-1}(x)$ is one-one and onto
(b) $f^{-1}(x)$ is one-one and into
(c) Area bounded between asymptote of curve $f(x)$ and $f^{-1}(x)$ is 9 sq units
(d) $f^{-1}(x)=\dfrac{2x-1}{1-x}$

Passage II

Directions (Q. Nos. 19-20) A box contains b red balls, $2b$ white balls and $3b$ blue balls, where b is a positive integer, 3 balls are selected at random from the box. If balls are drawn without replacement and A denotes the event that 'No two of the selected balls have the same colours, and B denotes the event that 'No two of the 3 drawn balls are blue.

19. The correct statement is/are
(a) There is exactly one value of b for which $P(A)=0.3$ and this value is less than 5
(b) There is more than one value of b for which $P(A)=0.3$
(c) $P(B)=\dfrac{2}{3}$, if $b=2$
(d) $P(B)=\dfrac{1}{2}$ for all value of b

20. If $P(A)=0.3$, then
(a) $P\left(\dfrac{A}{B}\right)=\dfrac{1}{2}$ (b) $P\left(\dfrac{B}{A}\right)=1$
(c) $P\left(\dfrac{A}{B}\right)=\dfrac{3}{5}$ (d) $P(A\cup B)=\dfrac{1}{2}$

Answers

Paper I

1. (1)	**2.** (2)	**3.** (7)	**4.** (7)	**5.** (5)	**6.** (1)	**7.** (1)	**8.** (4)	**9.** (a,b,c)	**10.** (a,b,c)
11. (c,d)	**12.** (b,c)	**13.** (a,c,d)	**14.** (b,c)	**15.** (a,b,d)	**16.** (b,c)	**17.** (b,c)	**18.** (b,c,d)		

19. A → q; B → r; C → s; D → p **20.** A → r; B → p; C → q; D → s

Paper II

1.(0)	**2.** (7)	**3.** (9)	**4.** (6)	**5.** (1)	**6.** (9)	**7.** (6)	**8.** (3)	**9.** (a,b,d)	**10.** (a,b,c,d)
11. (b,c,d)	**12.** (a,b,d)	**13.** (a,b,c,d)	**14.** (a,b,c)	**15.** (a,c,d)	**16.** (a,b,c)	**17.** (b,c)	**18.** (b,c,d)	**19.** (a,d)	**20.** (b,c,d)

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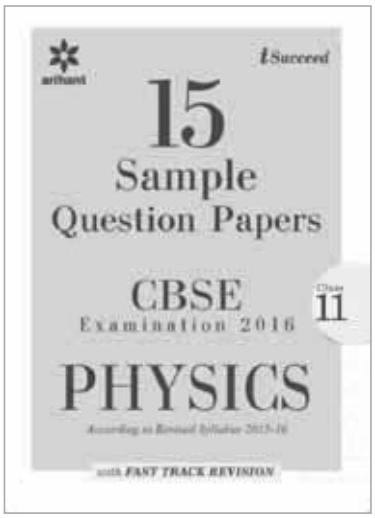

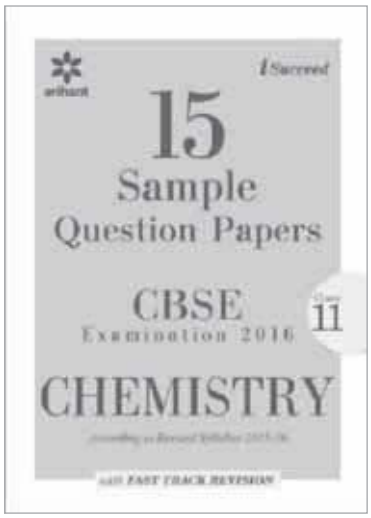

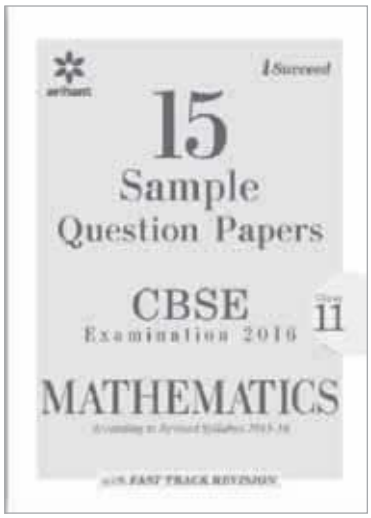

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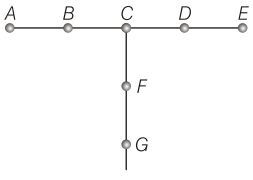
Alok Mani Tripathi

Single Integer Answer Type Questions on PERMUTATION AND COMBINATION

A collection of chapterwise best questions of their types

1. If the number of different circles passing through the points A, B, C, D, E, F and G as shown in the adjacent figure is n, then find $\frac{3n}{8}$.



2. Let $n_1 < n_2 < n_3 < n_4 < n_5$ be positive integers such that $n_1 + n_2 + n_3 + n_4 + n_5 = 20$. The number of such distinct arrangements $(n_1, n_2, n_3, n_4, n_5)$ is [JEE Advanced 2014]

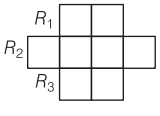
3. Let $n \geq 2$ be an integer. Take n distinct points on a circle and join each pair of points by a line segment. Colour the line segment joining every pair of adjacent points by blue and the rest by red. If the number of red and blue line segments are equal, then the value of n is [JEE Advanced 2014]

4. Five balls of different colours are to be placed in three boxes of different sizes. Each box can hold all five. If in n different ways we can place the balls so that no box remain empty , then find the value of $n/25$.

5. If the arithmetic mean of the smallest elements in all triplets of $A = \{1, 2, 3, 4, 5, 6\}$ is Z, then find the value of $4Z$.

6. If the number of ways in which the number 27720 can be split into two factors which are coprimes, is M, then what will be the value of $M^2/64$.

7. Six A have to be placed in the squares of figure below in such a way that each row contains atleast one A. If this can be done in Z different ways, then find $234/Z$.



8. If the number of ways, in which we can choose 3 squares on a chess board such that one of the squares has its two sides common to the other two squares, is W, then find $\frac{W}{73}$.

9. If the number of rectangles excluding squares from a rectangle of size 9×6, is N, then what will be the value of $\frac{N}{113}$?

10. Five distinct letters are to be transmitted through a communication channel. A total number of 15 blanks is to be inserted between the two letters with atleast three between every two.If the number of ways in which this can be done is X, then find $\left(\frac{X}{240} - 7\right)^2$.

11. In a plane there are two families of lines $y = x + r$, $y = -x + r$, where $r \in (0, 1, 2, 3, 4)$. Find the number of squares of diagonals of length 2 formed by the lines.

12. The number of four digit numbers, whose decimal notation contains not more than two distinct digits is x then $\frac{x+4}{290}$ is

13. If the number of non-negative integral solutions of $x_1 + x_2 + x_3 + 4x_4 = 20$ is x, then find $\frac{x+4}{90}$

14. If a, b, c, d are odd natural numbers such that $a + b + c + d = 20$, then X is the number of values of the ordered 4-tuple (a, b, c, d). Find $\frac{X}{33}$.

15. If the number of seven digit integers with sum of the digits equal to 10 and formed by using the digits 1, 2 and 3 only is I, then find the value of $I/11$.

16. The letters of the word COCHIN are permuted and all the permutations are arranged in an alphabetical order as in an English dictionary. If the number of words that appear before the word COCHIN is M, then $\left(\frac{M}{48}\right)^2$ is equal to [IIT JEE 2007]

17. It is given that r, s, t are prime numbers and p, q are the positive integers such that LCM of p, q is $r^2 s^4 t^2$. If Y is the number of ordered pairs (p, q), then find $\left|\frac{\sqrt{Y}}{5}\right|$. [IIT JEE 2006]

18. There are 4 oranges, 5 apples and 6 mangoes in a fruit basket. If number of ways a person can make a selection of fruits from among the fruits in the basket when all fruits of the same type are identical, is X, then find $(X/19 - 7)$.

19. Number of permutations of 1, 2, 3, 4, 5, 6, 7, 8 and 9 taken all at a time are such that the digit
1 appearing somewhere to the left of 2.
3 appearing to the left of 4 and 5 somewhere to the left of 6, is $k \times 7!$, then the value of k is .....

20. Numbers from 1 to 1000 are divisible by 60 but not by 24, is

21. If the number of positive integers from 1 to 1000, which are not divisible by 2, 3 or 5 is X, then find the value of $\frac{X}{133}$.

22. If the number of ways of dividing 15 men and 15 women into 15 couples, each consisting of a man and a woman, is X, then find the value of $\frac{X}{320}$.

23. If total number of positive integral solution of $15 < x_1 + x_2 + x_3 \le 20$, is X, then find the value of $\left(\frac{2X}{191} - 9\right)$.

24. In a polygon, no three diagonals are concurrent. If the total number of points of intersection of diagonals interior to the polygon is 70, and the number of diagonals of the polygon is K then $K/5$ is

25. A man has three friends. If the number of ways he can invite one friend everyday for dinner on six successive nights so that no friend is invited more than three times is Z, then $\left(\frac{Z}{170}\right)^2$ is equal to

26. $ABCD$ is a convex quadrilateral and 3, 4, 5, and 6 points are marked on the sides AB, BC, CD and DA, respectively. If the number of triangles with vertices on different sides is N, then find the value of $\frac{N}{38}$.

27. The last digit of $(1! + 2! + 3! + \ldots + 2005!)^{500}$ is equal to

28. If number of ways in which ₹18 can be distributed amongst four persons such that nobody receives less than ₹3 is X, then $\left|\frac{\sqrt{X+16}}{2}\right|$ is equal to

Answer *with* Explanations

1. *(9)* Since, we know that from three non-collinear points one and only one circle can be drawn.
Hence, the number of different circles
= Number of 3 non-collinear points selected from the given figure
$= {}^5C_2 \times {}^2C_1 + {}^4C_1 \times {}^2C_2 = 20 + 4 = 24$
Now, $\frac{3x}{8} = \frac{3 \times 24}{8} = 9$

Alternative Method
Number of different circles =
Number of selection of three non-collinear points
$= {}^7C_3 - {}^5C_3 - {}^3C_3 = 35 - 10 - 1 = 24$

2. *(7)* **Plan** Reducing the equation to a newer equation, where sum of variables is less. Thus, finding the number of arrangements becomes easier.
As, $n_1 \ge 1, n_2 \ge 2, n_3 \ge 3, n_4 \ge 4, n_5 \ge 5$
Let $n_1 - 1 = x_1 \ge 0, n_2 - 2 = x_2 \ge 0, \ldots, n_5 - 5 = x_5 \ge 0$
$\Rightarrow$ New equation will be
$x_1 + 1 + x_2 + 2 + \ldots + x_5 + 5 = 20$
$\Rightarrow x_1 + x_2 + x_3 + x_4 + x_5 = 20 - 15 = 5$
Now, $x_1 \le x_2 \le x_3 \le x_4 \le x_5$

x_1	x_2	x_3	x_4	x_5
0	0	0	0	5
0	0	0	1	4
0	0	0	2	3
0	0	1	1	3
0	0	1	2	2
0	1	1	1	2
1	1	1	1	1

So, 7 possible cases will be there.

3. *(5)* **Plan** Number of line segment joining pair of adjacent point = n Number of line segment obtained joining n points on a circle = ${}^nC_2$
Number of red line segments = ${}^nC_2 - n$
Number of blue line segments = n
$\therefore \quad {}^nC_2 - n = n \Rightarrow \frac{n(n-1)}{2} = 2n \Rightarrow n = 5$

4. *(6)* We have five balls of different colours and three boxes of different sizes sufficient to hold all five balls.
$\therefore$ Number of ways in which balls could be distributed, so that none is empty are (2, 2, 1) or (1, 1, 3)

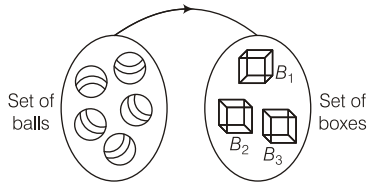
$= ({}^5C_2 \times {}^3C_2 \times {}^1C_1 + {}^5C_3 \times {}^2C_1 \times {}^1C_1) \times 3!/2!$

$= (10 \times 3 \times 1 + 10 \times 2 \times 1) \times 3 = 150$

$\Rightarrow \quad n = 150 \Rightarrow \frac{n}{25} = 6$

Alternative Method

The above problem can also be solved in the following way, which is easier than the first method. Actually the problem reduces to all possible onto functions from the set of five balls of different colours to the set of three boxes of different sizes.



Hence, the total number of required ways (n)

$= 3^5 - {}^3C_1 \cdot 2^5 + {}^3C_2 \cdot 1^5$

$= 243 - 96 + 3 = 246 - 96 = 150 \Rightarrow \frac{n}{25} = 6$

5. ***(7)*** Obviously the smallest elements in these triplets are 1, 2, 3, and 4. We shall find the number of triplets $n(1), n(2), n(3), n(4)$ respectively having 1, 2, 3, 4 as the smallest elements. For 1 to be smallest element, we have to select any 2 elements from the remaining 5 elements 2, 3, 4, 5, 6, so that $n(1) = {}^5C_2$. For 2 to be the smallest element, we have to select 2 elements from 4 higher numbers 3, 4, 5, 6 in ${}^4C_2$ ways, so that $n(2) = {}^4C_2$. Similarly, $n(3) = {}^3C_2$ and $n(4) = {}^2C_2$.

Augmented sum of all smallest numbers

$= 1 \times {}^5C_2 + 2 \times {}^4C_2 + 3 \times {}^3C_2 + 4 \times {}^2C_2$

$= 10 + 12 + 9 + 4 = 35$

and the total number of these smallest number

$= {}^5C_2 + {}^4C_2 + {}^3C_2 + {}^2C_2 = 10 + 6 + 3 + 1 = 20$

Hence, the AM required $= \frac{35}{20} = \frac{7}{4} \Rightarrow 4Z = 7$

6. ***(4)*** We have, $N = 27720 = 2^3 \times 3^2 \times 5 \times 7 \times 11$

Here, coprimes are 2, 3, 5, 7, 11.

Also, number of different factor in 27720 = 5

Hence, number of ways in which a composite number 27720 can be resolved into two factors which are relative prime

$= 2^{5-1} = 2^4 = 16 = M$

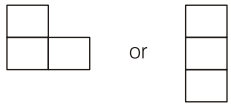
$\therefore \quad \frac{M^2}{64} = \frac{16 \times 16}{64} = 4$

7. ***(9)*** Since, all the A's are identical. So, we have to find selection of 6 squares from 8 squares, so that no row remains empty. Here, R_1 has 2 squares, R_2 has 4 squares and R_3 has 2 squares. The selection scheme for Z will be as

R_1	R_2	R_3	Number of selections
1	4	1	${}^2C_1 \times {}^4C_4 \times {}^2C_1 = 4$
1	3	2	${}^2C_1 \times {}^4C_3 \times {}^2C_2 = 8$
2	3	1	${}^2C_2 \times {}^4C_3 \times {}^2C_1 = 8$
2	2	2	${}^2C_2 \times {}^4C_2 \times {}^2C_2 = 6$

$\therefore \quad Z = 4 + 8 + 8 + 6 = 26 \Rightarrow 234/Z = \frac{234}{26} = 9$

8. ***(4)*** Either we have to choose



(i) Every square of (2 by 2)

will contribute four

shaped figure by removing anyone square out of it.
Number of ways to choose these square = $7 \times 7 = 49$.
So, total

shaped figure = $49 \times 4 = 196$

(ii) In every line, it is possible to have 6

shaped figure.

So, total number of these kind of figure = $6 \times 8 \times 2 = 96$

$\therefore$ Total number of cases (W) = 196 + 96 = 292

So, $\frac{W}{73} = \frac{292}{73} = 4$

9. ***(7)*** Here, $n = 6, P = 9$

$\therefore$ Number of rectangles excluding square (N)

$= \frac{6 \times 9}{4}(6+1)(9+1) - \sum_{r=1}^{6}(7-r)(10-r)$

$= 945 - \sum_{r=1}^{6}(70 - 17r + r^2) = 945 - 154 = 791 \Rightarrow \frac{N}{113} = \frac{791}{113} = 7$

10. ***(9)*** For $1 \le i \le 4$, let x_i (≥ 3) be the number of blanks between ith and $(i+1)$th letters. Then,

$x_1 + x_2 + x_3 + x_4 = 15 \quad \ldots(i)$

The number of solutions of Eq. (i)

= Coefficient of t^{15} in $(t^3 + t^4 + \ldots)^4$

= Coefficient of t^3 in $(1-t)^{-4}$

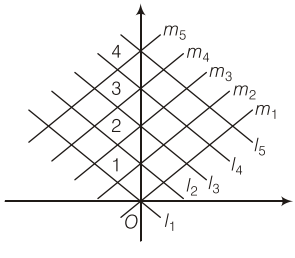
= Coefficient of t^3 in $[1 + {}^4C_1 t + {}^5C_2 t^2 + {}^6C_3 t^3 + \ldots]$

$= {}^6C_3 = 20$

But 5 letters can be Transmitted in 5! = 120 ways. Thus, the required number of arrangements = $120 \times 20 = 2400$

$\therefore \quad \frac{X}{240} = \frac{2400}{240} = 10 \Rightarrow \left(\frac{X}{240} - 7\right)^2 = (10-7)^2 = 9$

11. ***(9)*** There are two sets of five parallel lines at equal distances.



Clearly, lines like l_1, l_3, m_1, m_3 form a squares whose diagonal's length is 2. So, the number of required squares

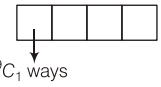
$= 3 \times 3 = 9$

[since, choices are $(l_1, l_3), (l_2, l_4), (l_3, l_5)$ for one set, etc.]

12. ***(2)*** We have to form four digit numbers whose decimal notation contains one or two different digits. Clearly, the number of four digit numbers with one digit such as 1111, 2222 = 9 etc. Let us consider two different digits.

Here, two cases arise

Case I Out of two digits, one digit is 0. We can select other non-zero digit in ${}^9C_1 = 9$ ways.



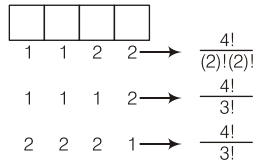
Let us consider the digits 1,0. Then, the four digits numbers are 1110, 1100, 1000 and their arrangements. When thousand place is 1, the remaining places can be arranged in $\left(\frac{3!}{2!} + \frac{3!}{2!} + 1\right)$

= 7 ways

$\therefore$ The number of required number = $7 \times 9 = 63$

Case II Two digits are non-two different digits. We can select two different non-zero digit out of 9 non-zero digits in ${}^9C_2 = 36$ ways.

Let us proceed with 1, 2



Then, possible combinations for four digits are 1, 1, 2, 2; 1, 1, 1, 2 and 2, 2, 2, 1 and their corresponding arrangements are $\frac{4!}{(2!)(2!)}$, $\frac{4!}{3!}$ and $\frac{4!}{3!}$, respectively.

Hence, the number of four digits numbers under given condition

$= {}^9C_2 (6 + 4 + 4) = 36 \times 14 = 504$

$\therefore$ Total number of four digit numbers whose decimal notation contains not more than two distinct digits

$= 9 + 63 + 504 = 576, \quad \frac{x+4}{290} = \frac{576+4}{290} = 2$

13. ***(6)*** The given equation is $x_1 + x_2 + x_3 + 4x_4 = 20$

Here, $x_1, x_2, x_3, x_4 \geq 0$

Keeping x_4 fixed, say k, i.e. $x_4 = k$, where $0 \leq k \leq 5$, the number of non-negative integral solution of the equation

$x_1 + x_2 + x_3 = 20 - 4x_4 = 20 - 4k$

is ${}^{20-4k+3-1}C_{3-1} = {}^{22-4k}C_2$, where $0 \leq k \leq 5$

Hence, the required number of non-negative integral solution of the given equation $x_1 + x_2 + x_3 + 4x_4 = 20$, x

$= \sum_{k=0}^{5} {}^{22-4k}C_2 = {}^{22}C_2 + {}^{18}C_2 + {}^{14}C_2 + {}^{10}C_2 + {}^{6}C_2 + {}^{2}C_2$

$= 231 + 153 + 91 + 45 + 15 + 1 = 536$

$\Rightarrow \quad \frac{x+4}{90} = \frac{536+4}{90} = \frac{540}{90} = 6$

14. ***(5)*** Let $a = 2p + 1, b = 2q + 1, c = 2r + 1, d = 2s + 1$, where p, q, r and s are non-negative integer.

$\therefore \; 2p + 1 + 2q + 1 + 2r + 1 + 2s + 1 = 20$

or $p + q + r + s = 8$

The required number of solutions (X)

= The number of non-negative integral solutions of $(p + q + r + s = 8)$

$= \frac{(8+3)(8+2)(8+1)}{6} = 165 \Rightarrow \frac{X}{33} = 5$

15. ***(7)*** There are two possible cases.

Case I Five 1's, one 2's, one 3's

Number of numbers = $\frac{7!}{5!} = 42$

Case II Four 1's, three 2's

Number of numbers = $\frac{7!}{4!3!} = 35$

Total number of numbers $(l) = 42 + 35 = 77$

$\therefore \quad \frac{l}{11} = \frac{77}{11} = 7$

16. ***(4)*** Total number of words starting with $C = 5! = 120$ which include the word COCHIN also.

$\therefore$ The number of words that appear before the word COCHIN is less than 120.

Now, total number of words starting with CC = 4! = 24 and similarly in each of the cases total words starting with

CH, CI, CN is 4! = 24

$\therefore$ Total number of words before starting with

$CO = 4 \times 24 = 96$

and the first word with CO in alphabetical order is COCHIN.

$\therefore \quad M = 96 \Rightarrow \left(\frac{M}{48}\right)^2 = 4$

17. ***(3)*** Since, r, s and t are prime numbers.

$\therefore$ Selection of p and q are as under

p	q	Number of ways
r^0	r^2	1 way
r^1	r^2	1 way
r^2	r^0, r^1, r^2	3 way

$\therefore$ Total number of ways to select $r = 5$

s^0	s^4	1 way
s^1	s^4	1 way
s^2	s^4	1 way
s^3	s^4	1 way
s^4	s^0, s^1, s^2, s^3, s^4	5 ways

Total number of ways to select $s = 9$

Similarly, the number of ways to select $t = 5$

$\therefore$ Total number of ways = $5 \times 9 \times 5 = 225$

Hence, $Y = 225$

$\Rightarrow \quad \left|\frac{\sqrt{Y}}{5}\right| = \left|\frac{15}{5}\right| = 3$

18. **(4)** Zero or more oranges can be selected out of 4 identical oranges in $4 + 1 = 5$ ways. Zero or more apples can be selected out of 5 identical apples in $5 + 1 = 6$ ways.

Zero or more mangoes can be selected out of 6 identical mangoes in $6 + 1 = 7$ ways.

$\therefore$ Total number of selections when all the three types of fruits are selected $= 5 \times 6 \times 7 = 210$

But in one of these selections number of each type of fruit is zero and hence this selection must be excluded.

$\therefore$ Required number $(X) = 210 - 1 = 209$

Hence, $X/19 - 7 = 4$

19. **(9)** Number of digits are 9.

Select 2 places for the digits 1 and 2 in ${}^9C_2$ ways.

From the remaining 7 places select any two places for 3 and 4 in ${}^7C_2$ ways and from the remaining 5 places select any two for 5 and 6 in ${}^5C_2$ ways.

Now, the remaining 3 digits can be filled in 3! ways.

$\therefore$ Total ways $= {}^9C_2 \cdot {}^7C_2 \cdot {}^5C_2 \cdot 3!$

$$= \frac{9!}{2!7!} \times \frac{7!}{2!5!} \times \frac{5!}{2!3!} \times 3 \times 2 = 9 \cdot 7!$$

Hence, $K = \dfrac{9 \cdot 7!}{7!} = 9$

20. **(8)** Let $n(A)$ = number divisible by 60 are $(60, 120, \ldots 960) = 16$

$\therefore$ $n(B)$ = number divisible by 24 = $(24, 48, \ldots, 984) = 41$

$n(A \cap B)$ = number divisible by both 24 and 60

$= 120, 240, \ldots, 960 = 8$

Hence, $n(A \cap \bar{B}) = n(A) - n(A \cap B) = 16 - 8 = 8$

21. **(2)** Let A_k be the set of positive integers from 1 to 1000, which is divisible by K, where $K = 2, 3$ or 5. Obviously, we have to find $n(A_2 \cup A_3 \cup A_5)$.

If [.] denotes the greatest integer function,

then $$n(A_2) = \left[\frac{1000}{2}\right] = 500, \; n(A_3) = \left[\frac{1000}{3}\right] = 333$$

$$n(A_5) = \left[\frac{1000}{5}\right] = 200$$

Here, $n(A_2 \cap A_3) = 166, n(A_3 \cap A_5) = 66,$

$n(A_2 \cap A_5) = 100, n(A_2 \cap A_3 \cap A_5) = 33$

Hence, $n(A_2 \cup A_3 \cup A_5) = 500 + 333 + 200 - 166 - 66 - 100 + 33 = 734$

Note that the number of positive integers from 1 to 1000 which are not divisible by 2, 3 or 5 is

$1000 - n(A_2 \cup A_3 \cup A_5) = 266$

$\therefore$ $X = 266 \Rightarrow \dfrac{X}{133} = 2$

22. **(4)** The number of ways of choosing first couple is ${}^{15}C_1 \cdot {}^{15}C_1 = 15^2$, then number of ways of choosing second couple is $({}^{14}C_1 \cdot {}^{14}C_1) = 14^2$ and so on.

Thus, the number of ways of choosing couples is

$$15^2 + 14^2 + 13^2 + \ldots + 1^2 = \frac{15 \times (15+1)(2 \times 15 + 1)}{6} = 1240$$

$\therefore$ $X = 1240 \Rightarrow \dfrac{X}{310} = 4$

23. **(1)** $15 < x_1 + x_2 + x_3 \le 20$

$\Rightarrow$ $x_1 + x_2 + x_3 = 16 + r, \quad r = 0, 1, 2, 3, 4.$

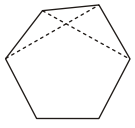
Now, number of positive integral solutions of $x_1 + x_2 + x_3 = 16 + r$ is ${}^{16+r+3-1}C_{3-1}$ i.e. ${}^{18+r}C_2$

Thus, $X = \sum_{r=0}^{4} {}^{18+r}C_2 = {}^{18}C_2 + {}^{19}C_2 + {}^{20}C_2 + {}^{21}C_2 + {}^{22}C_2$

$$= \frac{1}{2}(18 \times 17 + 19 \times 18 + 20 \times 19 + 21 \times 20 + 22 \times 21) = 955$$

$\Rightarrow$ $\dfrac{2X}{191} - 9 = 10 - 9 = 1$

24. **(4)** Let the number of sides be n. A selection of four vertices of the polygon gives, an interior intersection.



$\therefore$ ${}^nC_4 = 70$

$\Rightarrow$ $n(n-1)(n-2)(n-3)$

$= 24 \times 70 = 8 \times 7 \times 6 \times 5 \Rightarrow n = 8$

Hence, the number of diagonals $= {}^nC_2 - n$

$K = {}^8C_2 - 8 = 28 - 8 = 20 \Rightarrow K/5 = 20/5 = 4$

25. **(9)** Let x, y and z be the friends and a, b, c denote the case when x is invited a times, y is invited b times and z is invited c times. Now, we have the following possibilities

$(a, b, c) = (1, 2, 3)$ or $(3, 3, 0)$ or $(2\ 2\ 2)$

[grouping of 6 days of week]

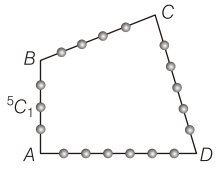
Hence, the total number of ways (Z) is

$$\frac{6!}{1!2!3!}3! + \frac{6!}{3!3!2!}3! + \frac{6!}{(2!2!2!)3!}3! = 360 + 60 + 90 = 510$$

$\therefore$ $\left(\dfrac{Z}{170}\right)^2 = 9$

26. **(9)** The number of triangles with vertices on sides AB, BC, CD is

${}^3C_1 \times {}^4C_1 \times {}^5C_1$.



Similarly, for other cases, the total number of triangles (N) is

${}^3C_1 \times {}^4C_1 \times {}^5C_1 + {}^3C_1 \times {}^4C_1 \times {}^6C_1 + {}^3C_1 \times {}^5C_1 \times {}^6C_1 + {}^4C_1 \times {}^5C_1 \times {}^6C_1$

$\Rightarrow$ $N = 342 \Rightarrow \dfrac{N}{38} = 9$

27. **(1)** Let $N = 1! + 2! + \ldots + 2005!$

$= (1! + 2! + 3! + 4!) + (5! + \ldots + 2005!)$

$= 33 +$ an integer having 0 in its unit's place

$=$ an integer having 3 in its unit's place

Hence, N^{500} is an integer having 1 in its unit's place.

28. **(5)** Suppose ith person receives ₹ x_i; $i = 1, 2, 3, 4$

Then, $x_1 + x_2 + x_3 + x_4 = 18$, where $x_i \ge 4$

Let $y_i = x_i - 3, i = 1, 2, 3, 4.$ Then,

$y_1 + y_2 + y_3 + y_4 = 6$

The total number of ways is equal to number of solutions of the above equation, which is given by

${}^{6+4-1}C_{4-1} = {}^9C_3 = 84 \Rightarrow X = 84$

$\Rightarrow$ $\left|\dfrac{\sqrt{X+16}}{2}\right| = \dfrac{\sqrt{84+16}}{2} = \dfrac{\sqrt{100}}{2} = \dfrac{10}{2} = 5$

SINE RULE

The sides of the triangle are proportional to the sine of the angle opposite to them in ΔABC

i.e. $\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}=2R$

where, R is radius of circumcircle of the triangle.

COSINE RULE

If the length of the three sides of triangle are known, then we can find all the angles by using cosine rule, which are given as

$$\cos A=\frac{b^2+c^2-a^2}{2bc},\ \cos B=\frac{c^2+a^2-b^2}{2ac}$$

and $\cos C=\frac{a^2+b^2-c^2}{2ab}$

PROJECTION FORMULA

In ΔABC, projection of AB on BC is BD.

$\therefore \quad BD=c\cos B$

Projection of AC on BC is CD.

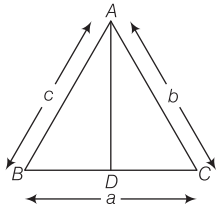
$\therefore \quad CD=b\cos C$

Now, $BC=BD+DC$

$\Rightarrow \quad a=c\cos B+b\cos C$

Similarly, $b=a\cos C+c\cos A$

$c=b\cos A+a\cos B$



Trigonometric Ratios of Half Angle

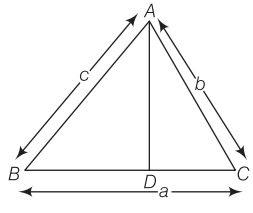
In ΔABC, if a, b and c are sides opposite to the vertices A, B, C respectively, then semi-perimeter, $s=\frac{a+b+c}{2}$, then

- $\sin\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{bc}}$
- $\sin\frac{B}{2}=\sqrt{\frac{(s-c)(s-a)}{ac}}$
- $\sin\frac{C}{2}=\sqrt{\frac{(s-a)(s-b)}{ab}}$
- $\cos\frac{A}{2}=\sqrt{\frac{s(s-a)}{bc}}$
- $\cos\frac{B}{2}=\sqrt{\frac{s(s-b)}{ac}}$
- $\cos\frac{C}{2}=\sqrt{\frac{s(s-c)}{ab}}$
- $\tan\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$
- $\tan\frac{B}{2}=\sqrt{\frac{(s-a)(s-c)}{s(s-b)}}$
- $\tan\frac{C}{2}=\sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$

AREA OF TRIANGLE

(i) Area of $\Delta ABC=\frac{1}{2}\times\text{Base}\times\text{Height}$

$$=\frac{1}{2}\times BC\times AD=\frac{1}{2}a(c\sin B)=\frac{1}{2}ac\sin B$$



Similarly,

$$\text{Area of }\Delta ABC=\frac{1}{2}ab\sin C=\frac{1}{2}bc\sin A$$

(ii) Area of $\Delta ABC=\sqrt{s(s-a)(s-b)(s-c)}$

NAPIER'S FORMULA

Napier's formula are also known as tangent rules. These are useful in calculating the remaining part of triangle, when two sides and included angle are given. These rules are as follows

(i) $\tan\left(\frac{B-C}{2}\right)=\frac{b-c}{b+c}\cot\frac{A}{2}$

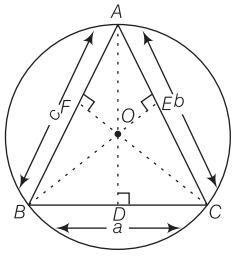
(ii) $\tan\left(\frac{C-A}{2}\right)=\frac{c-a}{c+a}\cot\frac{B}{2}$

(iii) $\tan\left(\frac{A-B}{2}\right)=\frac{a-b}{a+b}\cot\frac{C}{2}$

CIRCUMCIRCLE OF TRIANGLE

A circle passing through the vertices of triangle is called the circumcircle of the triangle and its centre is called circumcentre which lies on the point of intersection of the perpendicular bisectors of sides of the triangle. The radius of the circumcircle is called circumradius of the triangle and it is denoted by R.

$$R=\frac{a}{2\sin A}=\frac{b}{2\sin B}=\frac{c}{2\sin C}=\frac{abc}{4\Delta}$$



Note

- Distance of the circumcentre of the triangle can be find as follow from the sides of the triangle.
  $OD=R\cos A, OE=R\cos B, OF=R\cos C$
- Circumcentre of obtuse angled triangle lies outside the triangle.
- Circumcentre of right angled triangle is mid-point of the hypotenuse of the triangle.
- Circumcentre of acute angled triangle lies inside the triangle.

INCIRCLE OF THE TRIANGLE

The circle which can be inscribed in the triangle, is called incircle of the triangle. Incentre is basically the point of intersection of internal bisector of the angles of the triangle. Its radius is called inradius and usually denoted by r. Some important formulae is given below

(i) $r=\frac{\Delta}{s}$

(ii) $r=\begin{cases}(s-a)\tan A/2\\(s-b)\tan B/2\\(s-c)\tan C/2\end{cases}$

(iii) $r=\begin{cases}\frac{a\sin B/2\sin C/2}{\cos A/2}\\\frac{b\sin A/2\sin C/2}{\cos B/2}\\\frac{c\sin A/2\sin B/2}{\cos C/2}\end{cases}$

(iv) $r=4R\sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)$

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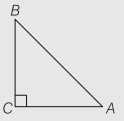
SOLUTION O

There are six elements of triangle, in which three are its s triangle are given, atleast one of which is its side, then ot triangle.

SOLUTION OF RIGHT ANGLED TRIANGLE

Let ΔACB be a given triangle with right angled at C, then



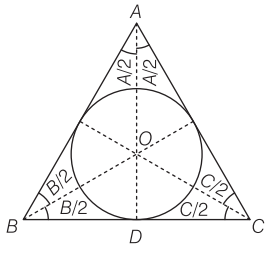
(i) **Solution** When two sides are given

Given elements	To be calculated
a, b	$\tan A=\frac{a}{b}, B=90°-A,$ $c=\frac{a}{\sin A}$ or $c=\sqrt{a^2+b^2}$
a, c	$\sin A=\frac{a}{c}, B=90°-A,$ $b=c\cos A$ or $b=\sqrt{c^2-a^2}$

(ii) **Solution** When one side and acute angle are given

Given elements	To be calculated
a, A	$B=90°-A, b=a\cot A,$ $c=\frac{a}{\sin A}$
c, A	$B=90°-A, a=c\sin A,$ $b=c\cot A$

S.No.
(i)
(ii)
(iii)
(iv)



ESCRIBED CIRCLE OF A TRIANGLE

The circle which touches the side BC and two sides AB and AC produced of ΔABC is called escribed circle opposite to angle A and its radius is denoted by r_1.

on through PT MAP

earn quickly...

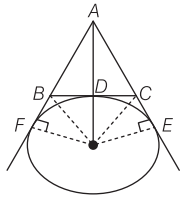
F TRIANGLE

sides and other three are its angle. If three elements of a her elements can be calculated. This is called solution of

SOLUTION OF OBLIQUE TRIANGLE

Given elements	To be calculated
Three sides a, b, c	$\sin A = \frac{2\Delta}{bc}, \sin B = \frac{2\Delta}{ac}, \sin C = \frac{2\Delta}{ab}$
Two sides and included angles are given	$\tan\left(\frac{A-B}{2}\right) = \frac{a-b}{a+b}\cot\frac{C}{2}; c = \frac{a\sin C}{\sin A}$ $\tan\left(\frac{B-C}{2}\right) = \frac{b-c}{b+c}\cot\frac{A}{2}; a = \frac{b\sin A}{\sin B}$ $\tan\left(\frac{C-A}{2}\right) = \frac{c-a}{c+a}\cot\frac{B}{2}; b = \frac{c\sin B}{\sin C}$
One side 'a' and two angles A and B	$b = \frac{c\sin B}{\sin C}, c = \frac{a\sin C}{\sin A}$
Two sides a, b, and opposite angle A	$\sin B = \frac{b}{a}\sin A, c = \frac{a\sin C}{\sin A}$ Now, different cases arise here If A is an acute angle and $a < b\sin A$, then $\sin B = \frac{b}{a}\sin A$ gives $\sin B > 1$ which is not possible, so no such triangles are possible.

Similarly, r_2, r_3 denote radii of escribed circle opposite to angles B and C respectively.



The centre of escribed circles are called the excentres. Following are important results on escribed circle

(i) $r_1 = \frac{\Delta}{s-a}, r_2 = \frac{\Delta}{s-b}, r_3 = \frac{\Delta}{s-c}$

(ii) $r_1 = s\tan\frac{A}{2}, r_2 = s\tan\frac{B}{2}, r_3 = s\tan\frac{C}{2}$

(iii) $r_1 = \frac{a\cos\frac{B}{2}\cos\frac{C}{2}}{\cos\frac{A}{2}}, r_2 = \frac{b\cos\frac{A}{2}\cos\frac{C}{2}}{\cos\frac{B}{2}}$,

$r_3 = \frac{c\cos\frac{A}{2}\cos\frac{B}{2}}{\cos\frac{C}{2}}$

(iv) $r_1 = 4R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$,

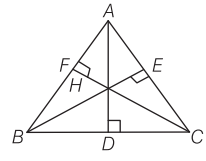
$r_2 = 4R\cos\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2}, r_3 = 4R\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}$

(v) $r_1 + r_2 + r_3 = 4R + r$

(vi) $r_1 r_2 + r_2 r_3 + r_3 r_1 = \frac{r_1 r_2 r_3}{r}$

ORTHOCENTRE OF A TRIANGLE

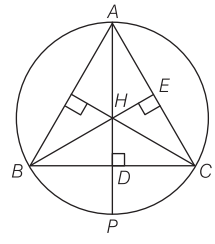
The orthocentre is the point of intersection of altitudes drawn from vertices of a triangle.



Note

- Orthocentre (H) of an acute angled triangle lies inside the triangle while in obtuse angled triangle it lies outside the triangle.
- Orthocentre (H) of right angled triangle lies at the vertex which contain right angle.
- Image of orthocentre (H) with respect to any side of the triangle lies on the circumcircle.

  $HD = DP \Rightarrow P$ is image of H with respect to line BC.



- Distance of orthocentre from vertices and sides of triangle are as follow

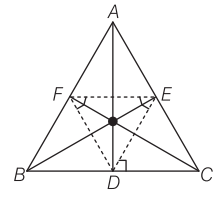
  $AH = 2R\cos A, DH = 2R\cos B\cos C$

  $BH = 2R\cos B, EH = 2R\cos A\cos C$

  $CH = 2R\cos C, FH = 2R\cos A\cos B$

PEDAL TRIANGLE

The triangle formed by joining the feet of the altitudes drawn from vertices to the opposite side is called pedal triangle.



ΔDEF shown in above figure is a pedal triangle.

Note

- In acute angled triangle, orthocentre of ΔABC is incentre of the pedal ΔDEF.
- Sides of pedal triangle in acute angled triangle are as follow $EF = a\cos A$, $DF = b\cos B$, $DE = c\cos C$
- Circumradius of pedal triangle (R') is

$$2R' = \frac{EF}{\sin(\angle EDF)} = \frac{a\cos A}{\sin(\pi - 2A)} = \frac{a}{2\sin A} = R$$

CENTROID OF TRIANGLE

The point of intersection of medians is called centroid of the triangle and it is denoted by G.

Note

- Centroid divides median in the ratio 2 : 1,

  i.e. $AG = \frac{2}{3}AD, BG = \frac{2}{3}BE, CG = \frac{2}{3}CF$

  $DG = \frac{1}{3}AD, GE = \frac{1}{3}BE, GF = \frac{1}{3}CF$

- Lengths of medians are

$$AD = \frac{1}{2}\sqrt{2b^2 + 2c^2 - a^2}, BE = \frac{1}{2}\sqrt{2a^2 + 2c^2 - b^2}$$

$$CF = \frac{1}{2}\sqrt{2a^2 + 2b^2 - c^2}$$

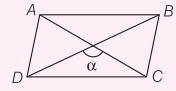
m-n Theorem

Let D be a point on the side BC of a ΔABC such that $BD : DC = m : n$ and $\angle ADC = \theta, \angle BAD = \alpha$ and $\angle DAC = \beta$, then (a) $(m+n)\cot\theta = m\cot\alpha - n\cot\beta$

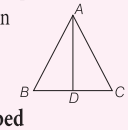
(b) $(m+n)\cot\theta = n\cot B - m\cot C$

Some Useful Results

- Area of quadrilateral $= \frac{1}{2} \times$ (Product of diagonals) $\times$ Side of included angle $= \frac{1}{2} \times BD \times AC\sin\alpha$



- Sum of opposite angles of cyclic quadrilateral is 180°.
- In cyclic quadrilateral sum of the product of opposite sides is equal to the product of diagonals. This is called Ptolemy's theorem.
- If sum of opposite sides of a quadrilateral is equal, then only circle can be inscribed in the quadrilateral.
- **Apollonius Theorem** If AD is median of the ΔABC, then $AB^2 + AC^2 = 2(AD^2 + BD^2)$



Radii of Inscribed and Circumscribed Circles of Regular Polygon

(i) $R = \frac{a}{2}\operatorname{cosec}\frac{\alpha}{2}$ (ii) $r = \frac{a}{2}\cot\frac{\pi}{n}$

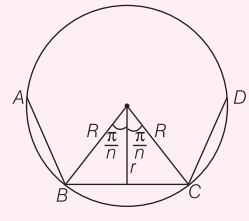
where a is length of each side and n is the number of sides or regular polygon.

(iii) Area of polygon

$$= \frac{1}{4}na^2\cot\frac{\pi}{n}$$

$$= nr^2\tan\frac{\pi}{n}$$

$$= \frac{n}{2}R^2\sin\left(\frac{2\pi}{n}\right)$$



@CLASS XII SYLLABUS

JEE Final Touch

Fast Track Revision

Indefinite Integrals

Integration is defined as a inverse process of differentiation. In this we find a function whose differential coefficient is given. If the differential coefficient of a function $F(x)$ is $f(x)$, i.e. if $\frac{d}{dx}(F(x)) = f(x)$, then $\int f(x)\,dx = F(x) + C$, where C is called the constant of integration and $F(x)$ is called anti-derivative or primitive or integral of function $f(x)$. Integration is very important topic for JEE main and Advanced point of view. Every year atleast one question is asked from this chapter.

ELEMENTARY INTEGRATION

Fundamental Integration Formulae

(i) $\int x^n dx = \frac{x^{n+1}}{n+1} + C \ (n \neq -1)$

(ii) $\int \frac{1}{x}\,dx = \log_e |x| + C$

(iii) $\int e^x dx = e^x + C$

(iv) $\int a^x dx = \frac{a^x}{\log_e a} + C = a^x \log_a e + C$

(v) $\int \sin x\,dx = -\cos x + C$

(vi) $\int \cos x\,dx = \sin x + C$

(vii) $\int \tan x\,dx = \log|\sec x| + C = -\log|\cos x| + C$

(viii) $\int \cot x\,dx = \log|\sin x| + C$

(ix) $\int \sec x\,dx = \log|\sec x + \tan x| + C$

$= -\log|\sec x - \tan x| + C = \log\left|\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right| + C$

(x) $\int \operatorname{cosec} x\,dx = -\log|\operatorname{cosec} x + \cot x| + C$

$= \log|\operatorname{cosec} x - \cot x| + C = \log\left|\tan\left(\frac{x}{2}\right)\right| + C$

(xi) $\int \sec x \tan x\,dx = \sec x + C$

(xii) $\int \operatorname{cosec} x \cot x\,dx = -\operatorname{cosec} x + C$

(xiii) $\int \sec^2 x\,dx = \tan x + C$

(xiv) $\int \operatorname{cosec}^2 x\,dx = -\cot x + C$

(xv) $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C$ or $-\cos^{-1} x + C$

(xvi) $\int \frac{dx}{1+x^2} = \tan^{-1} x + C$ or $-\cot^{-1} x + C$

(xvii) $\int \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1} x + C$ or $-\operatorname{cosec}^{-1} x + C$

Properties of Indefinite Integration

If $f(x)$ and $g(x)$ are two functions of a variable x, then

(i) $\frac{d}{dx}\left(\int f(x)dx\right) = f(x)$

(ii) $\int\left(\frac{d}{dx}f(x)\right)dx = f(x) + C$

(iii) $\int kf(x)dx = k\int f(x)dx$, where k is a constant

(iv) $\int \{f(x) \pm g(x)\}dx = \int f(x)dx \pm \int g(x)dx$

Note The above results can be generalised to the form

$\int \{k_1 f_1(x) \pm k_2 f_2(x) \pm k_3 f_3(x) \pm \ldots \pm k_n f_n(x)\}dx$

$= k_1\int f_1(x)dx \pm k_2\int f_2(x)dx \pm \ldots \pm k_n\int f_n(x)dx$

Some Standard Formulae

(i) $\int \frac{1}{x^2 + a^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C = -\frac{1}{a}\cot^{-1}\left(\frac{x}{a}\right) + C$

(ii) $\int \frac{1}{x^2 - a^2}\,dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + C\ (x > a)$

(iii) $\int \frac{1}{a^2 - x^2}\,dx = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + C\ (x < a)$

(iv) $\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right) + C = -\cos^{-1}\left(\frac{x}{a}\right) + C$

(v) $\int \frac{1}{\sqrt{x^2 + a^2}}\,dx = \log\left|x + \sqrt{x^2 + a^2}\right| + C$

(vi) $\int \frac{1}{\sqrt{x^2 - a^2}}\,dx = \log\left|x + \sqrt{x^2 - a^2}\right| + C$

(vii) $\int \frac{1}{x\sqrt{x^2 - a^2}}\,dx = \frac{1}{a}\sec^{-1}\left(\frac{x}{a}\right) + C = -\frac{1}{a}\operatorname{cosec}^{-1}\left(\frac{x}{a}\right) + C$

(viii) $\int \sqrt{a^2 - x^2}\,dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C$

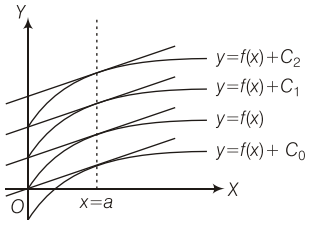
(ix) $\int \sqrt{x^2 + a^2}\,dx = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2}\log\left|x + \sqrt{x^2 + a^2}\right| + C$

(x) $\int \sqrt{x^2 - a^2}\,dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\log\left|x + \sqrt{x^2 - a^2}\right| + C$

Note If $\int f(x)\,dx = F(x) + C$, then $\int f(cx + d)\,dx = \frac{F(cx + d)}{c} + C$

Geometrical Interpretation of Indefinite Integral

Geometrically, the statement $\int f(x)dx = F(x) + C = y$ (say)



represents a family of curves. The different values of C correspond to different members of this family and these members can be obtained by shifting any one of the curves parallel to itself. Also, the tangents to the curves at the points of intersection of a line $x = a$ with the curves are parallel.

MATHS MAXIMA

1. Anti-derivative of a continuous function is differentiable.
2. If integrand is discontinuous at $x = x_1$, then its anti-derivative at $x = x_1$ need not be discontinuous.
3. Anti-derivative of a periodic function need not be a periodic function.

Best Practice SHOTS

1. The value of $\int \frac{2^{x+1} - 5^{x-1}}{10^x}\,dx$ is equal to

(a) $\frac{1}{5}\frac{2^{-x}}{\log 2} - \frac{2}{5^x \log 5}$ (b) $\frac{1}{5}\frac{2^x}{\log 2} + \frac{2}{5^x \log 5}$

(c) $\frac{1}{5}\frac{2^x}{\log 2} + \frac{1}{2}\frac{5}{\log 5}$ (d) None of these

2. Let $x = f''(t)\cos t + f'(t)\sin t$ and $y = -f''(t)\sin t + f'(t)\cos t$. Then, $\int\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right]^{1/2} dt$ equals to

(a) $f'(t) + f''(t) + C$ (b) $f''(t) + f'''(t) + C$

(c) $f(t) + f''(t) + C$ (d) $f'(t) - f''(t) + C$

3. If $\int f(x)\cos x\,dx = \frac{1}{2}f^2(x) + C$, then $f(x)$ can be

(a) x (b) 1

(c) $\cos x$ (d) $\sin x$

4. The value of $\int([\{x\}])\,dx$, where $\{.\}$ and $[.]$ denotes fractional part of x and greatest integer function respectively, is equal to

(a) 0 (b) 1 (c) 2 (d) –1

5. If $f\left(\frac{3x-4}{3x+4}\right) = x + 2$, then $\int f(x)\,dx$ is equal to

(a) $e^{x+2}\log_e\left|\frac{3x-4}{3x+4}\right| + C$

(b) $-\frac{8}{3}\log_e|1 - x| + \frac{2}{3}x + C$

(c) $\frac{8}{3}\log_e|x - 1| + \frac{x}{3} + C$

(d) None of the above

6. Let $f(x)$ be a polynomial of degree three satisfying $f(0) = -1$ and $f(1) = 0$. Also, 0 is a stationary point of $f(x)$. If $f(x)$ does not have an extremum at $x = 0$, then the value of the $\int \frac{f(x)}{x^3 - 1}\,dx$ is

(a) $\frac{x^2}{2} + C$ (b) $x + C$

(c) $\frac{x^3}{6} + C$ (d) None of these

7. If $\int f(x)\,dx = f(x)$, then $\int \{f(x)\}^2\,dx$ is equal to

(a) $\frac{1}{2}f\{(x)\}^2$ (b) $\{f(x)\}^3$ (c) $\frac{|f(x)|^3}{3}$ (d) $\{f(x)\}^2$

INTEGRATION BY SUBSTITUTIONS

Substitution Method

If $g(x)$ is a continuously differentiable function, then to evaluate integrals of the type

$$I = \int f(g(x)) \cdot g'(x)\, dx$$

we substitute $g(x) = t$ and $g'(x)dx = dt$. This substitution reduces the integral to $\int f(t)\, dt$, which can be integrate easily.

After evaluating this integral we substitute the value of t in the result.

Note When integrand is the product of two factors such that one is the derivative of the other, i.e. $I = \int f(x)\, f'(x)\, dx$. Then, we put $f(x) = t$ to convert it into a standard integral.

SPEED BOOSTER

The following three forms are very useful to evaluate integral directly

(i) $\int [f(x)]^n f'(x)\, dx = \dfrac{[f(x)]^{n+1}}{n+1} + C$, where $n \neq -1$

(ii) $\int \dfrac{f'(x)}{f(x)} dx = \log|f(x)| + C$ (iii) $\int \dfrac{f'(x)}{\sqrt{f(x)}} dx = 2\sqrt{f(x)} + C$

Integration Using Trigonometric Identities

When the integrand involves some trigonometric functions, then some known trigonometric identities are used to solve integral easily.

Important Trigonometric Identities

Some useful trigonometric identities are given below

1. $\sin 2x = 2\sin x \cos x = \dfrac{2\tan x}{1 + \tan^2 x}$
2. $\cos 2x = \cos^2 x - \sin^2 x = \dfrac{1 - \tan^2 x}{1 + \tan^2 x}$
   $= 2\cos^2 x - 1 = 1 - 2\sin^2 x$
3. $\tan 2x = \dfrac{2\tan x}{1 - \tan^2 x}$
4. $\cot 2x = \dfrac{\cot^2 x - 1}{2\cot x}$
5. $\sin 3x = -\sin^3 x + 3\cos^2 x \sin x = -4\sin^3 x + 3\sin x$
6. $\cos 3x = \cos^3 x - 3\sin^2 x \cos x = 4\cos^3 x - 3\cos x$
7. $\tan 3x = \dfrac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}$
8. $\cot 3x = \dfrac{3\cot x - \cot^3 x}{1 - 3\cot^2 x}$
9. $\tan\dfrac{x}{2} = \operatorname{cosec} x - \cot x = \sqrt{\dfrac{1-\cos x}{1+\cos x}} = \dfrac{\sin x}{1+\cos x} = \dfrac{1-\cos x}{\sin x}$
10. $\cot\dfrac{x}{2} = \operatorname{cosec} x + \cot x = \sqrt{\dfrac{1+\cos x}{1-\cos x}} = \dfrac{\sin x}{1-\cos x} = \dfrac{1+\cos x}{\sin x}$
11. $\tan(x + y) = \dfrac{\tan x + \tan y}{1 - \tan x \tan y}$
12. $\tan(x - y) = \dfrac{\tan x - \tan y}{1 + \tan x \tan y}$
13. $2\sin A \cos B = \sin(A + B) + \sin(A - B)$
14. $2\cos A \sin B = \sin(A + B) - \sin(A - B)$
15. $2\cos A \cos B = \cos(A + B) + \cos(A - B)$
16. $2\sin A \sin B = \cos(A - B) - \cos(A + B)$
17. $\sin A + \sin B = 2\sin\dfrac{A+B}{2} \cdot \cos\dfrac{A-B}{2}$
18. $\sin A - \sin B = 2\cos\dfrac{A+B}{2} \cdot \sin\dfrac{A-B}{2}$
19. $\cos A + \cos B = 2\cos\dfrac{A+B}{2} \cdot \cos\dfrac{A-B}{2}$
20. $\cos A - \cos B = -2\sin\dfrac{A+B}{2} \cdot \sin\dfrac{A-B}{2}$
21. Inverse trigonometric functions
    (i) $\sin^{-1}(\sin x) = x$ (ii) $\cos^{-1}(\cos x) = x$
    (iii) $\tan^{-1}(\tan x) = x$ (iv) $\operatorname{cosec}^{-1}(\operatorname{cosec} x) = x$
    (v) $\sec^{-1}(\sec x) = x$ (vi) $\cot^{-1}(\cot x) = x$

SPEED BOOSTER

Some useful conversion are given below

(i) $\sin^2 nx = \dfrac{1 - \cos 2nx}{2}$ (ii) $\cos^2 nx = \dfrac{1 + \cos 2nx}{2}$

(iii) $\sin nx = 2\sin\dfrac{nx}{2}\cos\dfrac{nx}{2}$

(iv) $\sin^3 nx = \dfrac{3}{4}\sin nx - \dfrac{1}{4}\sin 3nx$

(v) $\cos^3 nx = \dfrac{3}{4}\cos nx + \dfrac{1}{4}\cos 3nx$

Notice Board

IPU CET will be on May 8, 2016

Indraprastha University Common Entrance Test (IPU CET) 2016 will be conducted on May 8 (Sunday), 2016 for admissions to various UG engineering programmes, says a notification by the conducting authority. According to the notification released by Guru Gobind Singh Indraprastha University, IPU CET 2016 will be held between 2 pm to 4.30 pm on May 8, 2016 for admissions to B. Tech programmes, while the result will be declared latest by May 24, 2016. Candidates, who fulfill the eligibility criteria and wish to appear in IPU CET 2016, need to fill up the application form and submit it till March 29, 2016 (upto 4 pm).

Some Standard Substitutions

The following substitution are very useful to evaluate the integral involving the following expressions.

Standard Substitution

S. No.	Expression	Substitution
(i)	$(a^2 + x^2), \sqrt{x^2 + a^2}, \frac{1}{\sqrt{x^2 + a^2}}$	$x = a\tan\theta$ or $a\cot\theta$
(ii)	$(a^2 - x^2), \sqrt{a^2 - x^2}, \frac{1}{\sqrt{a^2 - x^2}}$	$x = a\sin\theta$ or $a\cos\theta$
(iii)	$(x^2 - a^2), \sqrt{x^2 - a^2}, \frac{1}{\sqrt{x^2 - a^2}}$	$x = a\sec\theta$ or $a\,\mathrm{cosec}\theta$
(iv)	$\frac{2x}{a^2 - x^2}, \frac{2x}{a^2 + x^2}, \frac{a^2 - x^2}{a^2 + x^2}$	$x = a\tan\theta$
(v)	$\sqrt{\frac{a - x}{a + x}}$ or $\sqrt{\frac{a + x}{a - x}}$	$x = a\cos 2\theta$
(vi)	$\sqrt{2ax - x^2}$	$x = a(1 - \cos\theta)$
(vii)	$\sqrt{\frac{x}{a + x}}, \sqrt{\frac{a + x}{x}}, \sqrt{x(a + x)}, \frac{1}{\sqrt{x(a + x)}}$	$x = a\tan^2\theta$ or $a\cot^2\theta$
(viii)	$\sqrt{\frac{x}{a - x}}; \sqrt{\frac{a - x}{x}}, \sqrt{x(a - x)}, \frac{1}{\sqrt{x(a - x)}}$	$x = a\sin^2\theta$ or $a\cos^2\theta$
(ix)	$\sqrt{\frac{x}{x - a}}; \sqrt{\frac{x - a}{x}}, \sqrt{x(x - a)}, \frac{1}{\sqrt{x(x - a)}}$	$x = a\sec^2\theta$ or $a\,\mathrm{cosec}^2\theta$
(x)	$\sqrt{\frac{x - \alpha}{\beta - x}}$ or $\sqrt{(x - \alpha)(\beta - x)}$	$x = \alpha\cos^2\theta + \beta\sin^2\theta$
(xi)	$(x \pm \sqrt{x^2 \pm a^2})^n$	expression inside the bracket $= t$
(xii)	$\frac{1}{(x + a)^{1 - \frac{1}{n}}(x + b)^{1 + \frac{1}{n}}}, (n \in N, n > 1)$	$\frac{x + a}{x + b} = t$

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8. The value of $\int \frac{\cos 7x - \cos 8x}{1 + 2\cos 5x}\,dx$ is equal to

(a) $\frac{\sin 2x}{2} + \frac{\cos 3x}{3} + C$ (b) $\sin x - \cos x + C$

(c) $\frac{\sin 2x}{2} - \frac{\cos 3x}{3} + C$ (d) None of these

9. If $\int \frac{(\sqrt{x})^5}{(\sqrt{x})^7 + x^6}\,dx = a\log\left(\frac{x^k}{1 + x^k}\right) + C$, then a and k are

(a) $\frac{2}{5}, \frac{5}{2}$ (b) $\frac{1}{5}, \frac{2}{5}$ (c) $\frac{5}{2}, \frac{1}{2}$ (d) $\frac{2}{5}, \frac{1}{2}$

10. Integral of $\sqrt{1 + 2\cot x\,(\cot x + \mathrm{cosec}x)}$ w.r.t. x, is

(a) $2\ln\cos\left(\frac{x}{2}\right) + C$ (b) $2\ln\sin\left(\frac{x}{2}\right) + C$

(c) $\frac{1}{2}\ln\cos\left(\frac{x}{2}\right) + C$ (d) $\ln\sin x - \ln(\mathrm{cosec}\,x - \cot x) + C$

11. Let $f(x) = \frac{x}{(1 + x^n)^{\frac{1}{n}}}$ for $n \geq 2$ and $g(x) = \underbrace{f_0 f_0 \ldots f}_{n \text{ times}}(x)$, then $\int x^{n-2}\,g(x)\,dx$ is equal to [IIT JEE 2007]

(a) $\frac{1}{n(n - 1)}(1 + nx^n)^{1 - \frac{1}{n}} + C$ (b) $\frac{1}{n - 1}(1 + nx^n)^{1 - \frac{1}{n}} + C$

(c) $\frac{1}{n(n + 1)}(1 + nx^n)^{1 + \frac{1}{n}} + C$ (d) $\frac{1}{n + 1}(1 + nx^n)^{1 + \frac{1}{n}} + C$

12. The evaluation of $\int \frac{px^{p+2q-1} - qx^{q-1}}{x^{2p+2q} + 2x^{p+q} + 1}\,dx$ is

(a) $-\left(\frac{x^p}{x^{p+q} + 1}\right) + C$ (b) $\frac{x^2}{x^{p+q} + 1} + C$

(c) $-\left(\frac{x^q}{x^{p+q} + 1}\right) + C$ (d) $\frac{x^p}{x^{p+q} + 1} + C$

13. If $I = \int \frac{dx}{\sec x + \mathrm{cosec}\,x}$, then I equals

(a) $\frac{1}{2}(\cos x + \sin x - \frac{1}{\sqrt{2}}\log|\mathrm{cosec}\,x - \cos x| + C$

(b) $\frac{1}{2}(\sin x - \cos x - \frac{1}{\sqrt{2}}\log|\mathrm{cosec}\,x + \cot x|) + C$

(c) $\frac{1}{\sqrt{2}}(\sin x + \cos x + \frac{1}{2}\log|\mathrm{cosec}\,x - \cos x|) + C$

(d) $\frac{1}{2}[\sin x - \cos x] - \frac{1}{2\sqrt{2}}\log\left|\mathrm{cosec}\left(x + \frac{\pi}{4}\right) - \cot\left(x + \frac{\pi}{4}\right)\right| + C$

14. If $xf(x) = 3f^2(x) + 2$, then $\int \frac{2x^2 - 12x\,f(x) + f(x)}{(6f(x) - x)(x^2 - f(x))^2}\,dx$ equals

(a) $\frac{1}{x^2 - f(x)} + C$ (b) $\frac{1}{x^2 + f(x)} + C$

(c) $\frac{1}{x - f(x)} + C$ (d) $\frac{1}{x + f(x)} + C$

15. $\int x\left(\frac{\ln a^{a^{\frac{x}{2}}}}{3a^{\frac{5x}{2}}b^{3x}}+\frac{\ln b^{b^x}}{2a^{2x}b^{4x}}\right)dx$, (where $a, b \in R^+$) is equal to

(a) $\frac{1}{6\ln a^2b^3}\cdot a^{2x}b^{3x}\cdot\ln\left(\frac{a^{2x}b^{3x}}{e}\right)+C$

(b) $\frac{1}{6\ln a^2b^3}\cdot\frac{1}{a^{2x}b^{3x}}\cdot\ln\left(\frac{1}{e\,a^{2x}b^{3x}}\right)+C$

(c) $\frac{1}{6\ln a^2b^3}\cdot\frac{1}{a^{2x}b^{3x}}\cdot\ln(a^{2x}b^{3x})+C$

(d) $\frac{-1}{6\ln a^2b^3}\cdot\frac{1}{a^{2x}b^{3x}}\cdot\ln(a^{2x}b^{3x})+C$

16. If $I^r(x)$ means log log log ... (x), the log being repeated r times, then $\int [xI(x)I^2(x)I^3(x)\ldots I^r(x)]^{-1}\,dx$ is equal to

(a) $I^{r+1}(x)+C$ (b) $\frac{I^{r+1}(x)}{r+1}+C$

(c) $I^r(x)+C$ (d) None of these

17. If $\int\frac{3e^x-5e^{-x}}{4e^x+5e^{-x}}dx = ax + b\ln(4e^x+5e^{-x})+C$, then

(a) $a=-\frac{1}{8}, b=\frac{7}{8}$ (b) $a=\frac{1}{8}, b=\frac{7}{8}$

(c) $a=-\frac{1}{8}, b=-\frac{7}{8}$ (d) $a=\frac{1}{8}, b=-\frac{7}{8}$

18. $\int\frac{2\sin x}{(3+\sin 2x)}dx$ is equal to

(a) $\frac{1}{2}\ln\left|\frac{2+\sin x-\cos x}{2-\sin x+\cos x}\right|-\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\sin x+\cos x}{\sqrt{2}}\right)+C$

(b) $\frac{1}{2}\ln\left|\frac{2+\sin x-\cos x}{2-\sin x+\cos x}\right|-\frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{\sin x+\cos x}{\sqrt{2}}\right)+C$

(c) $\frac{1}{4}\ln\left|\frac{2+\sin x-\cos x}{2-\sin x+\cos x}\right|-\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\sin x+\cos x}{\sqrt{2}}\right)+C$

(d) None of the above

19. The value of $\int\frac{(1-\cos\theta)^{\frac{2}{7}}}{(1+\cos\theta)^{\frac{9}{7}}}d\theta$ is

(a) $\frac{7}{11}\left(\tan\frac{\theta}{2}\right)^{\frac{11}{7}}+C$ (b) $\frac{7}{11}\left(\cos\frac{\theta}{2}\right)^{\frac{11}{7}}+C$

(c) $\frac{7}{11}\left(\sin\frac{\theta}{2}\right)^{\frac{11}{7}}+C$ (d) None of these

20. If $y=\int\frac{dx}{(1+x^2)^{\frac{3}{2}}}$ and $y=0$ when $x=0$, then the value of y when $x=1$ is

(a) $\frac{1}{\sqrt{2}}$ (b) $\sqrt{2}$

(c) $2\sqrt{2}$ (d) None of these

INTEGRATION BY PARTS

Theorem

If u and v are the differentiable function of x, then

$$\int(u\cdot v)dx = u\int v\,dx - \int\left[\left(\frac{d}{dx}u\right)\left(\int v\,dx\right)\right]dx$$

i.e. the integral of product of two functions

= (first function) × (integral of second function)

– integral of [(derivative of first function) × (integral of second function)]

MATHS MAXIMA

Following points should be remember while applying above rule

(i) If in the product of two functions, one of the functions is not directly integrable (e.g. $\log x$, $\sin^{-1}x$, $\cos^{-1}x$, $\tan^{-1}x$, ...), then we take it as the first function and the remaining function is taken as the second function.

(ii) Usually, we consider that function as Ist function, which comes first in the word ILATE, where I stands for inverse circular function, L stands for Logarithmic function, A stands for algebraic function, T stands for trigonometric function and E stands for exponential function.

(iii) For the integration of logarithmic or inverse trigonometric functions alone, take unity (1) as the second function.

(iv) If both the functions are directly integrable, then the first function is chosen in such a way that its derivative vanishes easily or the function obtained in integral sign in easily integrable.

SPEED BOOSTER

(i) $\int e^x\{f(x)+f'(x)\}\,dx = e^x f(x)+C$

(ii) $\int e^{ax}\sin(bx+c)\,dx = \frac{e^{ax}}{a^2+b^2}\{a\sin(bx+c)-b\cos(bx+c)\}+K$

(iii) $\int e^{ax}\cos(bx+c)\,dx = \frac{e^{ax}}{a^2+b^2}\{a\cos(bx+c)+b\sin(bx+c)\}+K$

(iv) $\int(xf'(x)+f(x))\,dx = x\,f(x)+C$

(v) $\int e^{ax}\left\{f(x)+\frac{f'(x)}{a}\right\}dx = e^{ax}\frac{f(x)}{a}+C$

(vi) $\int e^x\{f(x)-f''(x)\}\,dx = e^x(f(x)-f'(x))+C$

(vii) $\int e^x\{f(x)+f'''(x)\}\,dx = e^x\{f(x)-f'(x)+f''(x)\}+C$

(viii) $\int e^{f(x)}(g(x)f'(x)+g'(x))\,dx = e^{f(x)}g(x)+C$

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21. $\int \cot^{-1}(1 - x + x^2)\,dx = x\tan^{-1}x - \frac{1}{2}\log(1 + x^2) + A$, then A is equal to

(a) $-(1-x)\tan^{-1}(1-x^2) + \frac{1}{2}\log\{1 + (1-x^2)\} + C$

(b) $(1-x)\tan^{-1}(1-x^2) + \frac{1}{2}\log\{1 + (1-x^2)\} + C$

(c) $-(1-x)\tan^{-1}(1-x) + \frac{1}{2}\log\{1 + (1-x)^2\} + C$

(d) None of the above

22. $\int e^{ax}\cos(bx + c)\,dx$ is equal to

(a) $\frac{e^{ax}}{b^2}\{a\cos(bx + c) + b\sin(bx + c)\} + C$

(b) $\frac{e^{ax}}{a^2}\{a\cos(bx + c) + b\sin(bx + c)\} + C$

(c) $\frac{e^{ax}}{a^2 + b^2}\{a\cos(bx + c) + b\sin(bx + c)\} + C$

(d) None of the above

23. $\int x^2 e^x \sin x\,dx$ is equal to

(a) $\frac{e^x}{2}x^2[\sin x - \cos x] + C$

(b) $e^x[(x^2 - 1)\sin x - (x-1)^2\cos x] + C$

(c) $\frac{e^x}{2}[(x-1)^2\sin x - (x-1)^2\cos x] + C$

(d) $\frac{e^x}{2}[(x^2 - 1)\sin x - (x-1)^2\cos x] + C$

24. $\int \frac{1}{(x^2 + a^2)^2}\,dx$ is equal to

(a) $\frac{x}{2a^2(x^2 + a^2)} + \frac{1}{2a^3}\tan^{-1}\frac{x}{a} + C$

(b) $\frac{x}{x^2 + a^2} + \frac{1}{2a^3}\tan^{-1}\left(\frac{x}{a}\right) + C$

(c) $\frac{x}{2a^2(x^2 + a^2)} + \frac{1}{a}\tan^{-1}\frac{x}{a} + C$

(d) None of the above

25. $\int \frac{\cos^2 x + \sin 2x}{(2\cos x - \sin x)^2}\,dx$ is equal to

(a) $\frac{\cos x}{2\cos x - \sin x} - \frac{1}{5}x - \frac{2}{5}\log|2\cos x - \sin x| + C$

(b) $\frac{\sin x}{2\sin x - \cos x} - \frac{1}{5}x - \frac{2}{5}\log|2\cos x - \sin x| + C$

(c) $\frac{\cos x + \sin x}{2\sin x - \cos x} - \frac{1}{5}x - \frac{2}{5}\log|2\sin x - \cos x| + C$

(d) None of the above

26. $\int e^{2ax}\frac{1 - \cos 2ax}{1 + \sin 2ax}\,dx$ is equal to

(a) $-\frac{1}{a}e^{2ax}\cos\left(\frac{\pi}{4} + ax\right) + C$

(b) $-\frac{1}{2a}e^{2ax}\cot\left(\frac{\pi}{4} + ax\right) + C$

(c) $-\frac{1}{2a}e^{2ax}\cos\left(\frac{\pi}{4} + ax\right) + C$

(d) $-\frac{1}{a}e^{2ax}\operatorname{cosec}\left(\frac{\pi}{4} + ax\right) + C$

27. $\int \sqrt{x + a\sqrt{ax - a^2}}\,dx, 0 < a < 2 = \frac{2}{a^{3/2}}\{ax + a\sqrt{ax - a^2}\}^{\frac{3}{2}} - \frac{\sqrt{a}}{2}[A + B] + C$. Then,

(a) $A = \left[\left\{\left(\sqrt{ax - a^2} + \frac{a^2}{2}\right) + \sqrt{ax + a\sqrt{ax - a^2}}\right\}\right]$

(b) $B = \log\left[\left\{\left(\sqrt{ax - a^2} + \frac{a^2}{2}\right) + \sqrt{ax + a\sqrt{ax - a^2}}\right\}\right]$

(c) $A = \left[\left\{\left(\sqrt{ax - a^2} + \frac{a^2}{2}\right)\sqrt{ax + a\sqrt{ax - a^2}}\right\}\right]$

(d) $B = \log\left[\left\{\sqrt{ax - a^2} + \frac{a^2}{2}\sqrt{ax + a\sqrt{ax - a^2}}\right\}\right]$

INTEGRATION OF RATIONAL FUNCTION USING PARTIAL FRACTIONS

If a function is of the form $\frac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ $(\neq 0)$ are polynomials of degree m and n respectively.

(i) If $m < n$, this fraction is proper fraction and

(ii) If $m \geq n$, this fraction is improper fraction.

To convert an improper fraction into a proper fraction we divide $P(x)$ by $Q(x)$ such that $\frac{P(x)}{Q(x)} = M(x) + \frac{R(x)}{Q(x)}$ where $M(x)$ and $R(x)$ are polynomials and degree of $R(x)$ is less than degree of $Q(x)$. Any proper fractions can be expressed as the sum of rational function, if each having a simple factor of denominator.

Each such fraction is called partial fraction and process to obtained them is called decomposition of proper rational fractions and it depends on nature of the factors of a denominator.

Here, some proper fractions and their corresponding partial fractions are given below

S. No.	Types of proper rational functions	Partial fractions
(i)	$\frac{px+q}{(x-a)(x-b)}$, $a \neq b$	$\frac{A}{x-a}+\frac{B}{x-b}$
(ii)	$\frac{px^2+qx+r}{(x-a)(x-b)(x-c)}$, $a \neq b \neq c$	$\frac{A}{x-a}+\frac{B}{x-b}+\frac{C}{x-c}$
(iii)	$\frac{px+q}{(x-a)^3}$	$\frac{A}{x-a}+\frac{B}{(x-a)^2}+\frac{C}{(x-a)^3}$
(iv)	$\frac{px^2+qx+r}{(x-a)^2(x-b)}$	$\frac{A}{x-a}+\frac{B}{(x-a)^2}+\frac{C}{(x-b)}$
(v)	$\frac{px^2+qx+r}{(x-a)(x^2+bx+c)}$, where x^2+bx+c cannot be factorised.	$\frac{A}{x-a}+\frac{Bx+C}{x^2+bx+c}$
(vi)	$\frac{px^3+qx^2+rx+s}{(x^2+ax+b)(x^2+cx+d)}$, where (x^2+ax+b) and (x^2+cx+d) cannot be factorised	$\frac{Ax+B}{x^2+ax+b}+\frac{Cx+D}{x^2+cx+d}$

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28. $\int 3\sqrt{\tan x}\,dx = -\frac{1}{2}\log\left|\tan^{2/3}x+1\right| + \frac{1}{4}\log\left|\tan^{\frac{4}{3}}x-\tan^{\frac{2}{3}}x+1\right| + A$. Then, A is equal to

(a) $\frac{\sqrt{3}}{2}\tan^{-1}\left\{\frac{2\tan^{\frac{2}{3}}x-\tan^{\frac{4}{3}}x}{\sqrt{3}}\right\}+C$

(b) $\frac{\sqrt{3}}{2}\tan^{-1}\left\{\frac{2\tan^{\frac{2}{3}}x-1}{\sqrt{3}}\right\}+C$

(c) $\frac{\sqrt{3}}{2}\tan^{-1}\left\{\frac{2\tan^{\frac{2}{3}}x-\tan^{\frac{4}{3}}x}{3}\right\}+C$

(d) None of the above

29. $\int\frac{x^3+3x+2}{(x^2+1)^2(x+1)}dx = \frac{1}{4}\log(x^2+1)+\frac{x}{x^2+1}+\frac{3}{2}\tan^{-1}x - A$. Then, A is equal to

(a) $\frac{3}{2}\log|x+1|+C$ (b) $\frac{2}{3}\log|x+1|+C$

(c) $\frac{1}{2}\log|x+1|+C$ (d) $\frac{1}{2}\log|x^2+1|+C$

30. $\int\frac{1}{(x-1)\sqrt{x^2+4}}dx$ is equal to

(a) $-\frac{1}{\sqrt{5}}\log\left|\frac{1}{x-1}+\frac{1}{5}+\sqrt{\frac{x^2+4}{(x-1)^2}}\right|+C$

(b) $-\frac{1}{\sqrt{5}}\log\left|\frac{1}{x-1}+\frac{1}{5}+\sqrt{\frac{x^2+4}{5(x-1)^2}}\right|+C$

(c) $-\frac{1}{\sqrt{5}}\log\left|\frac{1}{x-1}+\frac{1}{5}\sqrt{\frac{x^2+4}{(x-1)^2}}\right|+C$

(d) None of the above

31. $\int\frac{\sqrt{1+x^2}}{1-x^2}dx = -\frac{1}{2}\log\left|\frac{\sqrt{1+x^2}-\sqrt{2x}}{\sqrt{1+x^2}+\sqrt{2x}}\right| - A + C$. Then, A is equal to

(a) $\log\left|x+\sqrt{1+x^2}\right|$ (b) $\log\left|\sqrt{1+x^2}-x\right|$

(c) $\log\left|\sqrt{1-x^2}+x\right|$ (d) None of these

32. $\int\frac{x^2-8x+7}{(x^2-3x-10)^2}dx = P\log|x-5|+Q\cdot\frac{1}{x-5} - R\log|x+2|-5\cdot\frac{1}{x+2}+C$. Then,

(a) $P=-\frac{45}{98}$ (b) $Q=\frac{8}{49}$ (c) $R=\frac{15}{49}$ (d) All of these

33. $\int\frac{x(x-1)}{(x^2+1)(x+1)\sqrt{x^3+x^2+x}}dx = \frac{1}{2}\log\left|\frac{\sqrt{x+\frac{1}{x}+1}-1}{\sqrt{x+\frac{1}{x}+1}+1}\right| - A + C$. Then, the value of A is equal to

(a) $\cos^{-1}\sqrt{1+\frac{1}{x}}$ (b) $\tan^{-1}\sqrt{x+\frac{1}{x}+1}$

(c) $\cot^{-1}\sqrt{x+\frac{1}{x}}$ (d) $\sin^{-1}\sqrt{1+\frac{1}{x}}$

34. $\int\frac{x^2-1}{x\sqrt{(x^2+\alpha x+1)(x^2+\beta x+1)}}dx$ is equal to

(a) $\log\left\{\frac{\sqrt{x^2+\alpha x+1}+\sqrt{x^2+\beta x+1}}{\sqrt{x}}\right\}+C$

(b) $2\log\left\{\frac{\sqrt{x^2+\alpha x+1}-\sqrt{x^2+\beta x+1}}{\sqrt{x}}\right\}+C$

(c) $\log\{\sqrt{x^2+\alpha x+1}-\sqrt{x^2+\beta x+1}\}+C$

(d) None of the above

INTEGRATION OF RATIONAL, IRRATIONAL ALGEBRAIC AND TRANSCENDENTAL FUNCTION

The following table will help you, to evaluate integral of some particular Rational, Irrational algebraic and Transcendental function.

	Types of the integral	Approach to solve
1.	$\int \frac{dx}{ax^2+bx+c}$ or $\int \frac{dx}{\sqrt{ax^2+bx+c}}$	Express ax^2+bx+c as sum or difference of two squares, i.e. $ax^2+bx+c = a\left[\left(x+\frac{b}{2a}\right)^2 + \frac{4ac-b^2}{4a^2}\right]$
2.	$\int \frac{px+q}{ax^2+bx+c}dx$ or $\int \frac{px+q}{\sqrt{ax^2+bx+c}}dx$	Put $px+q = \lambda \cdot \frac{d}{dx}(ax^2+bx+c)+\mu$. Now find the values of λ and μ and then integrate.
3.	$\int \frac{p(x)}{ax^2+bx+c}dx$	When $p(x)$ is a polynomial of degree 2 or more carry out the dimension and express in the form $\frac{p(x)}{ax^2+bx+c} = Q(x) + \frac{R(x)}{ax^2+bx+c}$, where $R(x)$ is a linear expression or constant, then integral reduces to the form discussed above.
4.	$\int \frac{x^2+a^2}{x^4+kx^2+a^4}dx$ or $\int \frac{x^2-a^2}{x^4+kx^2+a^4}dx$	After dividing both numerator and denominator by x^2. Put $x-\frac{a^2}{x}=t$ or $x+\frac{a^2}{x}=t$
5.	$\int \frac{dx}{a+b\sin^2 x}$, $\int \frac{dx}{a+b\cos^2 x}$, $\int \frac{dx}{a\sin^2 x+b\cos^2 x}$, $\int \frac{dx}{a\sin^2 x+b\cos^2 x+c}$, $\int \frac{dx}{(a\sin x+b\cos x)^2}$	To evaluate these types of integral, use following steps. ***Step* I** Divide numerator and denominator by $\cos^2 x$. ***Step* II** Replace $\sec^2 x$ (if any) in denominator by $1+\tan^2 x$ ***Step* III** Put $\tan x = t \Rightarrow \sec^2 x\,dx = dt$, simplify.
6.	$\int \frac{dx}{a+b\sin x}$, $\int \frac{dx}{a+b\cos x}$, $\int \frac{dx}{a\sin x+b\cos x}$, $\int \frac{dx}{a\sin x+b\cos x+c}$	To evaluate these types of integral proceed as follows ***Step* I** Put $\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}} = \sin x, \cos x = \frac{1-\tan^2 x/2}{1+\tan^2 x/2}$ ***Step* II** Replace $1+\tan^2\frac{x}{2}$ by $\sec^2\frac{x}{2}$. ***Step* III** Put $\tan\frac{x}{2}=t$, simplify
7.	$\int \frac{a\sin x+b\cos x}{c\sin x+d\cos x}dx$	Put $a\sin x+b\cos x = \lambda\cdot\frac{d}{dx}(c\sin x+d\cos x)+\mu\,(c\sin x+d\cos x)$, then integral reduces to $\int \frac{a\sin x+b\cos x}{c\sin x+d\cos x}dx = \lambda\int\frac{c\cos x-d\sin x}{c\sin x+d\cos x}dx + \mu\int\frac{c\sin x+d\cos x}{c\sin x+d\cos x}dx$
8.	$\int \frac{a\sin x+b\cos x+c}{p\sin x+q\cos x+r}dx$	Put $a\sin x+b\cos x+c = \lambda\left\{\frac{d}{dx}(p\sin x+q\cos x+r)\right\} + \mu\,(p\sin x+q\cos x+r)+r$, then integral reduces to $\lambda\int\frac{p\cos x-q\sin x}{p\sin x+q\cos x+r}dx + \mu\int\frac{p\sin x+q\cos x+r}{p\sin x+q\cos x+r}dx + r\int\frac{dx}{p\sin x+q\cos x+r}$
9.	$\int\sqrt{\tan x}\,dx$, $\int\sqrt{\cot x}\,dx$, $\int\frac{dx}{\sin^4 x+\cos^4 x}$, $\int\frac{dx}{\sin^6 x+\cos^6 x}$, $\int\frac{\pm\sin x\pm\cos x}{a+b\sin x\cos x}dx$	Put $\tan x = t^2 \Rightarrow sec^2 x = 2t\,dt$, then integral reduces to the form 4.
10.	$\int\frac{dx}{(Ax+B)\sqrt{Cx+D}}$	Substitute $Cx+D=t^2$, then substitution will reduce the given integral into $\int\frac{2dt}{At^2-AD+BC}$

	Types of the integral	Approach to solve
11.	$\int \sin^n x \cos^n x\, dx$	To evaluate this type of integral, we use following substitution. (i) If m is an odd positive integer put $\cos x = t$ (ii) If n is an odd positive integer put $\sin x = t$ (iii) If m and n are both positive odd integers, put $\sin x = t$ or $\cos x = t$ (iv) If neither m nor n is an odd positive integer, then we look for their sum $m + n$. *(a)* If $m + n$ = negative even integer, then convert the given integrand in term of $\tan x$ and $\sec^2 x$ and then put $\tan x = t$. *(b)* If m and n are small even intergers, then convert them in term of multiple angles by using the appropriate formula *(c)* If m and n are large even positive integers, then change them in multiple angles with the help of Euler's formula in complex numbers.
12.	$\int \frac{dx}{Ax^2 + B\sqrt{Cx + D}}$	Substitute $Cx + D = t^2$, then substitution will reduce the given integral into $\int \frac{2C\, dt}{At^4 - 2DAt^2 + (AD^2 + BC^2)}$
13.	$\int \frac{dx}{(x-k)^r \sqrt{ax^2 + bx + c}}$	Substitute $x - k = \frac{1}{t}$, then the substitution will reduce the given integral into $\int \frac{t^{r-1}}{At^2 + Bt + C}\, dt$
14.	$\int \frac{dx}{(Ax^2 + B)\sqrt{Cx^2 + D}}$	Substitute $x = \frac{1}{t}$, then $\int \frac{dx}{(Ax^2 + B)\sqrt{Cx^2 + D}} = \int \frac{-\frac{1}{t^2}\, dt}{\left(\frac{A}{t^2} + B\right)\sqrt{\frac{C}{t^2} + D}}$ $= -\int \frac{t\, dt}{(A + Bt^2)\sqrt{C + Dt^2}}$. Now put $C + Dt^2 = u^2$ reduces is into the form $\int \frac{du}{u^2 \pm a^2}$
15.	$\int \frac{ax^2 + bx + c}{(dx + e)\sqrt{fx^2 + gx + h}}\, dx$	Put $ax^2 + bx + c = A_1 (dx + e) \frac{d}{dx}(fx^2 + gx + h) + B_1 (dx + e) + c_1$, where A_1, B_1, C_1 are constants.
16.	$\int x^m (a + bx^n)^p\, dx$ where m, n, p are rational numbers	***Case* I** If p is a positive integer, then integrand is expanded by binomial expansion. ***Case* II** If p is negative integer substitute $x = t^k$, where k is LCM of denominator of m and n. ***Case* III** $\frac{m+1}{n}$ is an integer, put $a + bx^n = t^\alpha$, where α is denominator of the fraction p. ***Case* IV** $\frac{m+1}{n} + p$ is an integer, put $a + bx^n = t^\alpha x^n$ where α is denominator of the fraction p.

Integration of Hyperbolic Function

(i) $\int \sinh x\, dx = \cosh x + C$

(ii) $\int \cosh x\, dx = \sinh x + C$

(iii) $\int \text{sech}^2 x\, dx = \tanh x + C$

(iv) $\int \text{cosech}^2 x\, dx = -\coth x + C$

(v) $\int \text{sech}\, x \tanh x\, dx = -\text{sech}\, x + C$

(vi) $\int \text{cosech}\, x \coth x\, dx = -\text{cosech}\, x + C$

3 new IIITs: Admission to begin from 2016-17

The Ministry of Human Resource Development (MHRD) will set up 3 new Indian Institutes of Information Technology (IIITs) in collaboration with private players. A memorandum of understanding (MoU), recently signed between MHRD and private players Tata Consultancy Services, Tata Motors, M/s ADCC Infocad, Nagpur and M/s Hubtown, Mumbai, confirmed this. According to the MoU, three new IIITs - Ranchi, Nagpur and Pune, would be set up in collaboration among the central government, state government and industry. All the three IIITs will have their own infrastructure, but as that will take some time to be in place, IIIT Ranchi, Pune and Nagpur will run from temporary campuses, for the academic session 2016-17. All the three IIITs will admit students through JEE Main 2016.

REDUCTION FORMULAE

(i) $\int \sin^n x dx = -\frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n}\int \sin^{n-2} x\, dx$

e.g. $\int \sin^3 x dx = -\frac{\sin^2 x \cos x}{3} + \frac{2}{3}\int \sin x\, dx$

$= -\frac{\sin^2 x \cos x}{3} - \frac{2}{3}\cos x + C$

(ii) $\int \cos^n x\, dx = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n}\int \cos^{n-2} x\, dx$

e.g. $\int \cos^4 x\, dx = \frac{\cos^3 x \sin x}{4} + \frac{3}{4}\int \cos^2 x\, dx$

$= \frac{\cos^3 x \sin x}{4} + \frac{3}{4}\left[\frac{\cos x \sin x}{2} + \frac{1}{2}x\right] + C$

$= \frac{1}{4}\cos^3 x \sin x + \frac{3}{8}\sin x \cos x + \frac{3}{8}x + C$

(iii) $\int \tan^n x\, dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x\, dx$

e.g. $\int \tan^3 x\, dx = \frac{\tan^2 x}{2} - \int \tan x\, dx = \frac{\tan^2 x}{2} + \log \cos x + C$

(iv) $\int \cot^n x\, dx = -\frac{\cot^{n-1} x}{n-1} - \int \cot^{n-2} x\, dx$

e.g. $\int \cot^4 x\, dx = -\frac{\cot^3 x}{3} - \int \cot^2 x\, dx = -\frac{\cot^3 x}{3} + \frac{\cot x}{1} + \int dx$

$= -\frac{\cot^3 x}{3} + \cot x + x + C$

(v) $\int \sec^n x\, dx = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{(n-2)}{(n-1)}\int \sec^{n-2} x\, dx$

e.g. $\int \sec^4 x\, dx = \frac{\sec^2 x \tan x}{3} + \frac{2}{3}\int \sec^2 x\, dx$

$= \frac{\sec^2 x \tan x}{3} + \frac{2}{3}(\tan x) + C$

Best Practice SHOTS

35. $\int \frac{\sin^3 x/2}{\cos\frac{x}{2}\sqrt{\cos^3 x + \cos^2 x + \cos x}}\, dx$ is equal to

(a) $\cos^{-1}\sqrt{\sec x + \tan x + 1} + C$

(b) $\tan^{-1}\sqrt{\sin x + \cos x + 1} + C$

(c) $\sin^{-1}\sqrt{\tan x + \sec x + 1} + C$

(d) $\tan^{-1}\sqrt{\cos x + \sec x + 1} + C$

36. $\int \frac{1}{x[6(\log x)^2 + 7\log x + 2]}\, dx$ is equal to

(a) $\log\left|\frac{2\log x + 1}{3\log x + 2}\right| + C$ (b) $\log\left|\frac{3\log x + 1}{2\log x + 2}\right| + C$

(c) $\log\left|\frac{\log x + 1}{\log x + 2}\right| + C$ (d) $\log\left|\frac{\log x + 3}{\log x + 1}\right| + C$

37. If $\int x\sqrt{1 + x - x^2}\, dx = -\frac{1}{3}(1 + x - x^2)^{\frac{3}{2}} + A + C$, then A is equal to

(a) $\frac{1}{2}\left[\frac{1}{2}\left(x - \frac{1}{2}\right)\sqrt{1 + x - x^2} + \frac{5}{8}\sin^{-1}\left(\frac{2x-1}{\sqrt{5}}\right)\right]$

(b) $\frac{1}{2}\left[\left(x - \frac{1}{2}\right)\sqrt{1 + x - x^2} + \sin^{-1}\left(\frac{2x-1}{\sqrt{5}}\right)\right]$

(c) $\frac{1}{2}\left[\left(x - \frac{1}{2}\right)\sqrt{1 + x - x^2} + \frac{1}{8}\sin^{-1}\left(\frac{2x-1}{\sqrt{5}/2}\right)\right]$

(d) None of the above

38. If $\int (x+1)\sqrt{1 - x - x^2}\, dx = -\frac{1}{3}(1 - x - x^2)^{\frac{3}{2}} + A + C$, then the value of A is

(a) $\sqrt{1 - x - x^2} + \frac{5}{16}\sin^{-1}\left(\frac{2x+1}{\sqrt{5}}\right)$

(b) $\frac{1}{8}\sqrt{1 - x - x^2} + \frac{\sqrt{5}}{2}\sin^{-1}\left(\frac{2x+1}{\sqrt{5}}\right)$

(c) $\frac{1}{8}(2x + 1)\sqrt{1 - x - x^2} + \frac{5}{16}\sin^{-1}\left(\frac{2x+1}{\sqrt{5}}\right)$

(d) None of the above

39. If $\frac{3\sin x + 2\cos x}{3\cos x + 2\sin x}\, dx = ax + b\log|3\cos x + 2\sin x| + C$, then

(a) $a = \frac{5}{13}, b = \frac{-12}{13}$ (b) $a = \frac{12}{13}, b = \frac{-5}{13}$

(c) $a = \frac{12}{13}, b = \frac{5}{13}$ (d) $a = \frac{-12}{5}, b = \frac{-5}{13}$

40. If $\int \frac{1}{(5 + 4\cos x)^2}\, dx$ is equal to

(a) $\frac{10}{9}\tan^{-1}\left(\frac{\tan\frac{x}{2}}{3}\right) + \frac{4}{9}\left(\frac{\sin x}{5 + 4\cos x}\right) + C$

(b) $\frac{10}{27}\tan^{-1}\left(\frac{\tan\frac{x}{2}}{3}\right) + \frac{4}{9}\left(\frac{\sin x}{5 + 4\cos x}\right) + C$

(c) $\frac{5}{9}\tan^{-1}\left(\frac{\tan\frac{x}{2}}{9 + \tan^2\frac{x}{2}}\right) + \frac{4}{9}\left(\frac{\sin x}{5 + 4\cos x}\right) + C$

(d) None of the above

41. The value of $\int \frac{x^2+5x+3}{x^2+3x+2}\,dx$ is

(a) $\frac{x}{2} + \log|x^2+3x+2| - 2\log\left|\frac{x+1}{x+2}\right| + C$

(b) $x + \log|x^2+3x+2| - 2\log\left|\frac{x+1}{x+2}\right| + C$

(c) $x + \log|x^2+3x+2| - \frac{1}{2}\log\left|\frac{x+1}{x+2}\right| + C$

(d) None of the above

42. $\int \frac{1+\sin x}{\sin x\,(1+\cos x)}\,dx$ is equal to

(a) $\frac{1}{2}\left[\log\left|\tan\frac{x}{2}\right| + \frac{\tan\frac{x}{2}}{2} + 2\tan\frac{x}{2}\right] + C$

(b) $\frac{1}{2}\left[\log\left|\tan\frac{x}{2}\right| + \frac{\tan^2\frac{x}{2}}{2} + 2\tan\frac{x}{2}\right] + C$

(c) $\frac{1}{2}\left[\log\left|\tan\frac{x}{2}\right| + \frac{\tan^2\frac{x}{2}}{4} + 2\tan\frac{x}{2}\right] + C$

(d) None of the above

43. If $\int \frac{x^2+2x+3}{\sqrt{x^2+x+1}}\,dx = \frac{(2x+5)}{4}\sqrt{x^2+x+1} + A + C$, then A is equal to

(a) $\frac{15}{8}\log\left|\left(x+\frac{5}{4}\right) - \sqrt{x^2+x+1}\right|$

(b) $\frac{5}{4}\log\left|\left(x+\frac{1}{2}\right) - \sqrt{x^2+x+1}\right|$

(c) $\frac{15}{8}\log\left|\left(x+\frac{1}{2}\right) + \sqrt{x^2+x+1}\right|$

(d) None of the above

44. If $\int \frac{\cos^2 x \sin x}{\sin x - \cos x}\,dx = \frac{1}{4}\log|\sin x - \cos x| + A + C$, then A is equal to

(a) $\frac{3}{8}x + \frac{1}{4}\cos 2x + \frac{1}{4}\sin 2x$

(b) $\frac{1}{5}\cos 2x + \frac{1}{8}\sin 2x$

(c) $\frac{1}{2}x + \frac{1}{4}\sin 2x + \frac{1}{8}\cos 2x$

(d) $\frac{3}{8}x + \frac{1}{4}\sin 2x + \frac{1}{8}\cos 2x$

45. If $\int \frac{3\cos x + 2}{\sin x + 2\cos x + 3}\,dx = \lambda x + \mu \log|\sin x + 2\cos x + 3| + A + C$. Then, A is equal to

(a) $-\frac{8}{5}\tan^{-1}\left(\frac{\tan x + 1}{2}\right)$

(b) $-\frac{8}{5}\tan^{-1}\left(\frac{\tan\frac{x}{2}+1}{2}\right)$

(c) $-\frac{8}{5}\tan^{-1}\left(\frac{\tan\frac{x}{2}}{2}\right)$

(d) None of the above

46. $\int \frac{1}{\sqrt{3}\sin x + \cos x}\,dx$ is equal to

(a) $\frac{1}{2}\log\left|\tan\left(\frac{x}{2}+\frac{\pi}{12}\right)\right| + C$

(b) $\frac{1}{2}\log\left|\sec\left(\frac{x}{2}+\frac{\pi}{12}\right)\right| + C$

(c) $\frac{1}{2}\log\left|\tan\left(\frac{x}{12}+\frac{\pi}{12}\right)\right| + C$

(d) None of the above

47. The value of $\int \frac{1}{1+\sin x + \cos x}\,dx$ is equal to

(a) $\log\left\{\tan\frac{x}{2} + 1\right\} + C$

(b) $\log\left\{\sec\frac{x}{2} + 1\right\} + C$

(c) $\log\left\{\frac{1}{1+\tan\frac{x}{2}} + 1\right\} + C$

(d) None of the above

48. $\int e^x\left[\frac{1+(1+x^2)\tan^{-1}x}{1+x^2}\right]dx$ is equal to

(a) $e^x \cdot \tan^{-1}x + C$ (b) $e^x \cdot \tan^{-1}\frac{x}{2} + C$

(c) $\frac{e^x}{1+x^2} + C$ (d) $-e^x\tan^{-1}x + C$

49. $\int e^{\sin^{-1}x}\left[\frac{x+\sqrt{1-x^2}}{\sqrt{1-x^2}}\right]dx$ is equal to

(a) $e^{\sin^{-1}x} + C$ (b) $xe^{\sin^{-1}x} + C$

(c) $-e^{\sin^{-1}x} + C$ (d) $-xe^{\sin^{-1}x} + C$

Answers

1. (a)	**2.** (c)	**3.** (d)	**4.** (a)	**5.** (b)	**6.** (b)	**7.** (a)	**8.** (d)	**9.** (a)	**10.** (b)
11. (a)	**12.** (c)	**13.** (d)	**14.** (a)	**15.** (b)	**16.** (a)	**17.** (a)	**18.** (c)	**19.** (a)	**20.** (a)
21. (c)	**22.** (c)	**23.** (d)	**24.** (a)	**25.** (a)	**26.** (b)	**27.** (c)	**28.** (b)	**29.** (c)	**30.** (b)
31. (a)	**32.** (d)	**33.** (b)	**34.** (a)	**35.** (d)	**36.** (a)	**37.** (a)	**38.** (c)	**39.** (b)	**40.** (b)
41. (b)	**42.** (b)	**43.** (c)	**44.** (b)	**45.** (b)	**46.** (a)	**47.** (a)	**48.** (a)	**49.** (b)	

Formula at a Glance

1. $\int x^n dx = \frac{x^{n+1}}{n+1} + C \ (n \in R - \{-1\})$

   If $n = -1$

   i.e. $\int \frac{1}{x} dx = \ln x + C$

2. $\int \cos x \, dx = \sin x + C$

3. $\int \sin x \, dx = -\cos x + C$

4. $\int \sec^2 x \, dx = \tan x + C$

5. $\int \text{cosec}^2 x \, dx = -\cot x + C$

6. $\int \sec x \cdot \tan x \, dx = \sec x + C$

7. $\int \text{cosec } x \cdot \cot x \, dx = -\text{cosec } x + C$

8. $\int \tan x \, dx = \log|\sec x| + C = -\log|\cos x| + C$

9. $\int \cot x \, dx = \log|\sin x| + C$

10. $\int \sec x \, dx = \log|\sec x + \tan x| + C = \log\left|\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right| + C$

11. $\int \text{cosec } x \, dx = \log|\text{cosec } x - \cot x| + C = \log\left|\tan\frac{x}{2}\right| + C$

12. $\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C = -\frac{1}{a}\cot^{-1}\left(\frac{x}{a}\right) + C$

13. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C = -\cos^{-1}\left(\frac{x}{a}\right) + C$

14. $\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a}\sec^{-1}\left(\frac{x}{a}\right) + C = -\frac{1}{a}\text{cosec}^{-1}\left(\frac{x}{a}\right) + C$

15. $\int a^x dx = \frac{a^x}{\log_e a}, a > 0 \left(Ex. \int e^x dx = \frac{e^x}{\log_e e} = e^x + C\right)$

Note If $\int f(x)dx = F(x) + C$, then $\int f(ax + b)\,dx = \frac{1}{a}F(ax + b) + C$

Fundamental Rule of Integration

1. $\int kf(x)dx = k\int f(x)dx$

2. $\int (f(x) \pm g(x))dx = \int f(x)dx \pm \int g(x)dx$

   In general $\int (f_1(x) \pm f_2(x) \pm f_3(x) \pm \ldots \pm f_n(x))dx$
   $= \int f_1(x)dx \pm \int f_2(x)dx \pm \ldots \pm \int f_n(x)dx$

Integration by Substitution

Basic rule of substitution

$\int f(x)dx = \int f\{\phi(z)\} \cdot \phi'(z)dz$

Substitute $x = \phi(z)$

Integration by Parts

If u and v be two functions of x, then $\int uvdx = u\int vdx - \int\left(\frac{du}{dx}\int vdx\right)dx$ is known as formula for integeration by parts.

1. In finding integrals by this method proper choice of u and v is essential. In general the function as u is taken which comes first in the word ILATE
   - I - Inverse circulated function
   - L - Logarithmic function
   - A - Alegbraic function
   - T - Trigonometric function
   - E - Exponential function

Note If both the functions are trigonometric, take that function as 'v' whose integral is simpler.

If both the functions are algebraic, take that function as u whose differential coefficient is simpler.

Integration of Rational Function

If integrand is of the form $\frac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are polynomials of x.

Case I If $\frac{P(x)}{Q(x)}$ is proper rational function (i.e. highest power of x in $Q(x) < P(x)$), integeration by partial fraction is done.

Case II If $\frac{P(x)}{Q(x)}$ is improper rational function (i.e. highest power of x in $Q(x) \geq P(x)$. Then, it is made proper rational function by simple division and then partial fraction is used, i.e.

$$\frac{P(x)}{Q(x)} = \frac{f(x)\cdot Q(x) + h(x)}{Q(x)} = f(x) + \frac{h(x)}{Q(x)}$$

$f(x)$ is integrated by general rule and $\frac{h(x)}{Q(x)}$ is integrated by partial fractions.

Integration of Certain Irrational Expressions

1. If the integrand is a rational function of fractional powers of an independent variable x i.e. the function $R\ (x, x^{p_1/q_1}, \ldots, x^{p_k/q_k})$. Substitute $x = t^m$, m is LCM of $q_1, q_2, \ldots, q_k$.

2. If the integrand is a rational function of x and fractional powers of a linear fractional function of the form $\frac{ax + b}{cx + d}$ i.e.

   $$R\left(x, x^{p_1/q_1}, \ldots, x^{p_k/q_k}, \left(\frac{ax + b}{cx + d}\right), \left(\frac{ax + b}{cx + d}\right)^{\alpha_1/\beta_1}, \ldots, \left(\frac{ax + b}{cx + d}\right)^{\alpha_n/\beta_n}\right),$$

   substitute $\frac{ax + b}{cx + d} = t^m$, where m is LCM of $\beta_1, \beta_2, \ldots, \beta_n$.

3. Integerals of the form $I = \int \frac{Mx + N}{\sqrt{ax^2 + bx + c}}dx$

   Substitute $x + \frac{b}{2a} = t$ and reduce it to the form $I = M_1\int \frac{tdt}{\sqrt{at^2 + k}} + N_1$, N_1 and k are new constants.

4. Integrals of the form $I = \int \frac{P_m(x)}{\sqrt{ax^2 + bx + c}}$.

   $P_m(x)$ is polynomial of degree m

$$I = P_{m-1}(x)\sqrt{ax^2 + bx + c} + k\int \frac{dx}{\sqrt{ax^2 + bx + c}},$$

where $P_{m-1}(x)$ is polynomial of degree $(m-1)$ and k is constant.

Coefficient of $P_{m-1}(x)$ is determined by the method of undetermined coefficients.

5. Integrals of the form $\int \frac{dx}{(x-a_1)^m \cdot \sqrt{ax^2 + bx + c}}$

Substitute $(x - a_1) = \frac{1}{t}$

6. For the integration of the form $\int \sqrt{ax^2 + bx + c}\, dx$ or $\int \frac{dx}{\sqrt{ax^2 + bx + c}}$ or $\int \frac{dx}{ax^2 + bx + c}$

1st convert $ax^2 + bx + c$ in the form $A^2 + x^2$, $x^2 - A^2$ or $A^2 - x^2$, where A is constant and x is $px + q$.

Following results is used

(i) $\int \frac{dx}{x^2 \pm a^2} = \log\left|x + \sqrt{x^2 \pm a^2}\right| + C$

(ii) $\int \sqrt{a^2 - x^2}dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + C$

(iii) $\int \sqrt{x^2 \pm a^2}\, dx = \frac{x}{2}\sqrt{x^2 \pm a^2} \pm \frac{a^2}{2}\log(x + \sqrt{x^2 \pm a^2}) + C$

(iv) $\int \frac{1}{x^2 - a^2} dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + C$

Integration of the Form of $\int x^m (a + bx^n)^p\, dx$, when m, n and p are rational numbers

***Case* I** If P is a positive integer, then integrand is expanded by binomial expansion.

***Case* II** If P is negative integer substitute $x = t^k$, where k is LCM of denominator of m and n.

***Case* III** $\frac{m+1}{n}$ is an integer. Put $a + bx^n = t^\alpha$, where α is denominator of the fraction P.

***Case* IV** $\frac{m+1}{n} + P$ is an integer, put $a + bx^n = t^\alpha x^n$, where α is denominator of the fraction P.

Integration of Trigonometric Function

1. Integration of the form $\int \sin^m x \cdot \cos^n x dx$

(a) m = odd positive integer, substitute $z = \cos x$.

(b) n = odd positive integer, substitute $z = \sin x$.

(c) If both m and n are odd positive integer put $z = \sin x$ or $\cos x$.

(d) m and n both are not odd positive integer, if

(i) $m + n$ is even negative integer put $z = \tan x$.

(ii) $(m + n)$ is even positive integer, express the integrands as the algebraic sum of sines and cosines of multiple angles using formula

$$\sin^2 x = \frac{1 - \cos 2x}{2}$$

$$\cos^2 x = \frac{1 + \cos 2x}{2}, \sin^3 x = \frac{3\sin x - \sin 3x}{4},$$

$$\cos^3 x = \frac{3\cos x + \cos 3x}{4}$$

2. Integral of the form $I_1 = \int \tan^m x \cdot \sec^n x\, dx$ or $I_2 = \int \cot^m x \cdot \operatorname{cosec}^n x\, dx$

(a) If n is even positive integer, then put $z = \tan x$ in I_1, $z = \cot x$ in I_2.

(b) If m is even positive integer, then put $(\sec^2 x - 1)$ in place of $\tan^2 x$ in I_1 and $(\operatorname{cosec}^2 x - 1)$ in place of $\cot^2 x$ in I_2 and substitute $z = \tan x$ (in I_1) $= \cot x$ (in I_2)

(c) $m = 0$, n is odd positive integer greater than 1, then method of integration of parts is used .

Detailed Solutions

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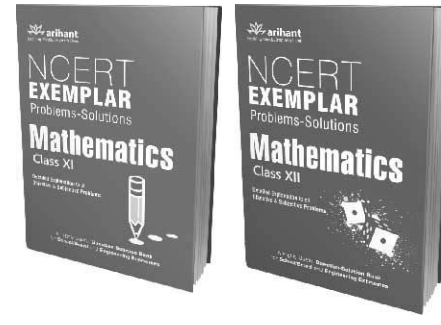

AVAILABLE FOR
XI & XII Physics, Chemistry, Maths & Bio
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Definite Integrals

Let 'f' be a function of x defined in the closed interval $[a, b]$ and F be another function, such that $F'(x) = f(x)$ for all x in the domain of 'f', then

$$\int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a)$$

is called definite integral of the function $f(x)$ over the interval $[a, b]$, a and b are called the limits of integration, a being the lower limit and b be the upper limit. The interval $[a, b]$ is called the interval of integration.

Note In definite integrals constant of integration never present.

MATHS MAXIMA

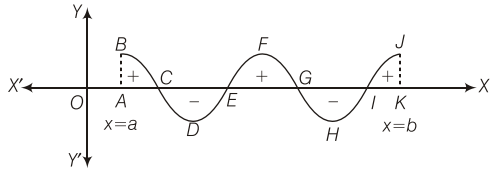
(i) If $f(x)$ is not defined at $x = a$ and $x = b$, but defined in open interval (a, b), then $\int_a^b f(x)\,dx$ can be evaluated.

(ii) In $\int_a^b f(x)\,dx$, the function f needs to be well defined and continuous in the closed interval $[a, b]$.

(iii) If $\int_a^b f(x)\,dx = 0$, then the equation $f(x) = 0$ has atleast one root lying in (a, b) provided f is a continuous function in (a, b).

Geometrical Interpretation of Definite Integral

- If $f(x) > 0$ for all $x \in [a, b]$, then $\int_a^b f(x)\,dx$ is numerically equal to the area bounded by the curve $y = f(x)$, the X-axis and the straight lines $x = a$, $x = b$.
- In general, $\int_a^b f(x)\,dx$ represents an algebraic sum of areas of the region bounded by the curve $y = f(x)$, the X-axis and the straight lines $x = a$ and $x = b$. The areas above X-axis are taken with plus sign and the areas below X-axis are taken with minus sign



i.e. $\int_a^b f(x)\,dx$ = ar (ABC) − ar (CDE) + ar (EFG) − ar (GHI) + ar (IJK)

BASIC PROPERTIES OF DEFINITE INTEGRALS

Evaluation of Definite Integral by Substitution

To evaluate integral of the form $\int_a^b f\{g(x)\}\,g'(x)\,dx$, we use following steps

Step I Put $g(x) = t \Rightarrow g'(x)\,dx = dt$

Step II Find the limits of integration for new variable, i.e. $g(a)$ and $g(b)$. The integral reduces to $\int_{g(a)}^{g(b)} f(t)\,dt$

Step III Integrate the definite integral obtained in step II, with help of suitable formula.

Caution The substitution must be continuous in the interval of integration.

Summation of Series by Definite Integral

Another way of describing $\int_a^b f(x)\,dx$ is that, it is a limiting case of summation of an infinite series, provided $f(x)$ is continuous on $[a, b]$, i.e. $\int_a^b f(x)\,dx = \lim_{n\to\infty} h\sum_{r=1}^{n-1} f(a + rh)$, where $h = \frac{b-a}{n}$. The converse is also true, i.e. if we have an infinite series of the above form, it can be expressed as definite integral. In order to express the infinite series as definite integral, we use following steps

Step I Express the given series in the form of $\lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{n} f\left(\frac{r}{n}\right)$

Step II Replace $\frac{r}{n}$ by x, $\frac{1}{n}$ by dx and $\lim_{n\to\infty}\Sigma$ by the sign of integration $(\int)$.

Step III The lower and upper limit of integration are limiting values of $\frac{r}{n}$ for the first and the last terms respectively,

i.e. lower limit $= \lim_{n\to\infty}\frac{r_1}{n}$, upper limit $= \lim_{n\to\infty}\frac{r_n}{n}$

where, r_1 and r_n are the least and greatest values of r, respectively.

Basic Properties of Definite Integrals

(i) $\int_a^b f(x)dx = \int_a^b f(t)dt$ [changing variable]

(ii) $\int_a^b f(x)dx = -\int_b^a f(x)dx$ [interchanging limit]

(iii) $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$, where c may lie inside the interval $[a, b]$. [splitting limits]

Note This property is useful when function is discontinuous or non-differentiable at $x = c$. This property can be generalized in the following form

$$\int_a^b f(x)\,dx = \int_a^{c_1} f(x)\,dx + \int_{c_1}^{c_2} f(x)\,dx + \int_{c_2}^{c_3} f(x)\,dx + \ldots + \int_{c_n}^{b} f(x)\,dx$$

(iv) $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

(v) $\int_a^{2a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(2a-x)dx$

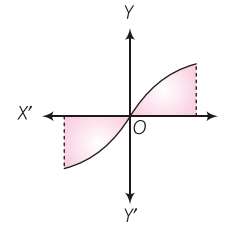
(vi) $\int_0^{2a} f(x)dx = \begin{cases} 0, \text{ if } f(2a-x) = -f(x) \\ 2\int_0^a f(x)dx, \text{ if } f(2a-x) = f(x) \end{cases}$

(vii) $\int_{-a}^{a} f(x)dx = \begin{cases} 0, \text{ if } f(x) \text{ is odd i.e. } f(-x) = -f(x) \\ 2\int_0^a f(x)\,dx, \text{ if } f(x) \text{ is even i.e. } f(-x) = f(x) \end{cases}$

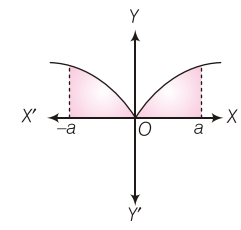
Graphically

$\int_{-a}^{a} f(x)dx = 0$, when $f(x)$ is symmetric about origin (Fig. 1) and

$\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$, when $f(x)$ is symmetric about Y-axis (Fig. 2)



(Fig. 1)



(Fig. 2)

REMEMBER

- $\int_0^{\pi/2} \log \sin x\,dx = \int_0^{\pi/2} \log \cos x\,dx = \frac{\pi}{2}\log\left(\frac{1}{2}\right) = -\frac{\pi}{2}\log(2)$
- $\int_0^{2a} f(x)dx = \int_0^a \{f(a-x) + f(a+x)\}\,dx$
- $\int_a^b f(x)dx = (b-a)\int_0^1 f[(b-a)x + a]dx$

Best Practice SHOTS

1. The value of $\int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}}\,dx$ is

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{2}$ (d) π

2. The value(s) of $\int_0^1 \frac{x^4(1-x)^4}{1+x^2}\,dx$ is/are [IIT JEE 2010]

(a) $\frac{22}{7} - \pi$ (b) $\frac{2}{105}$ (c) 0 (d) $\frac{71}{15} - \frac{3\pi}{2}$

3. The value of $\int_0^1 x\,(1-x)^{99}\,dx$ is

(a) $\frac{1}{10100}$ (b) $\frac{11}{10100}$ (c) $\frac{1}{10010}$ (d) None

4. Let $I = \int_0^1 \frac{\sin x}{\sqrt{x}}\,dx$ and $J = \int_0^1 \frac{\cos x}{\sqrt{x}}\,dx$. Then, which one of the following is true? [AIEEE 2008]

(a) $I > \frac{2}{3}$ and $J < 2$ (b) $I > \frac{2}{3}$ and $J > 2$

(c) $I < \frac{2}{3}$ and $J < 2$ (d) $I < \frac{2}{3}$ and $J > 2$

5. The value of $\int_{-5}^{5} |x+2|\,dx$ is

(a) 25 (b) 29 (c) 0 (d) 33

6. The value of $\int_{\pi/6}^{\pi/3} \frac{1}{1+\tan^3 x}\,dx$ is

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{6}$ (c) $\frac{\pi}{8}$ (d) $\frac{\pi}{12}$

7. The value of $\int_0^{\pi/2} \log\tan x\,dx$ is

(a) 0 (b) π (c) 2π (d) 3π

8. If $m = \int_{-2}^{0} \frac{|\sin x|}{\left[\frac{x}{\pi}\right] + \frac{1}{2}}\,dx$, $n = \int_0^2 \frac{|\sin x|}{\left[\frac{x}{\pi}\right] + \frac{1}{2}}\,dx$ (where, $[\cdot]$ = GIF), then

(a) $m = n$ (b) $m = -n$

(c) $m = 2n$ (d) None of these

9. The value of $\int_0^{\pi/2} \log(\sin x)\,dx = \int_0^{\pi/2} \log(\cos x)\,dx$ is

(a) $\frac{\pi}{2}\log 2$ (b) $-\frac{\pi}{2}\log 2$

(c) 0 (d) None of these

10. The value of $\int_0^{32\pi/3} \sqrt{1+\cos 2x}\,dx$ is

(a) $22\sqrt{2} + \frac{\sqrt{3}}{2}$ (b) $22\sqrt{2} - \frac{\sqrt{3}}{2}$

(c) $22\sqrt{2} - \sqrt{\frac{3}{2}}$ (d) None of these

11. The value of the integral $\int_0^{\infty} \frac{x\log x}{(1+x^2)^2}\,dx$ is

(a) 0 (b) 7

(c) $5\log 13$ (d) $2\log 5$

12. The value of $\int_0^{\infty} \log\left(1+\frac{1}{x}\right)\frac{dx}{1+x^2}$ is

(a) $\pi\log 2$ (b) $\log 2$

(c) 0 (d) None

13. The value of $\int_0^{\pi} \ln(1+\cos x)\,dx$ is

(a) $\frac{\pi}{2}\log 2$ (b) $\pi\log 2$ (c) $-\pi\ln 2$ (d) 0

14. The value of $\int_0^1 \frac{\ln x}{\sqrt{1-x^2}}\,dx$ is

(a) $\pi \ln 2$ (b) $\frac{-\pi}{2}\ln 2$

(c) $\frac{\pi}{2}\ln 2$ (d) None of these

15. The value $\int_0^{\pi} \frac{dx}{1+5^{\cos x}} + \int_{-2}^{2}\left(\frac{5-x}{5+x}\right)dx$ is

(a) π (b) $\frac{\pi}{2}$

(c) $\pi + \log 5$ (d) None of these

16. The value of $\int_0^a \frac{x^5\,dx}{(2a^2-x^2)^3}$ is

(a) $\frac{\log 2}{2} - \frac{1}{4}$ (b) $\frac{\log 2}{2} + \frac{1}{2}$

(c) $\frac{-\log 2}{2} - \frac{1}{2}$ (d) None of these

17. The value of $\int_a^b (x-a)^3 (b-x)^4\,dx$ is $\frac{(b-a)^m}{n}$. Then, (m, n) is

(a) (6, 260) (b) (8, 250) (c) (4, 240) (d) None

18. If $I(m, n) = \int_0^1 t^m (1+t)^n\,dt$, then the expression for $I(m, n)$ in terms of $I(m+1, n-1)$ is **[IIT JEE 2003]**

(a) $\frac{2^n}{m+1} - \frac{n}{m+1} I(m+1, n-1)$

(b) $\frac{2^n}{m+1} + \frac{n}{m+1} I(m+1, n-1)$

(c) $\frac{n}{m+1} I(m+1, n-1)$

(d) $\frac{m}{m+1} I(m+1, n-1)$

19. If $f(y) = e^y$, $g(y) = y$; $y > 0$ and $F(t) = \int_0^t f(t-y)\,g(y)\,dy$, then **[AIEEE 2003]**

(a) $F(t) = 1 - e^{-1}(1+t)$ (b) $F(t) = e^t - (1+t)$

(c) $F(t) = te^t$ (d) $F(t) = te^{-t}$

20. The value of the integral $I = \int_0^1 x\,(1-x)^n\,dx$ is **[AIEEE 2003]**

(a) $\frac{1}{n+1}$ (b) $\frac{1}{n+2}$

(c) $\frac{1}{n+1} - \frac{1}{n+2}$ (d) $\frac{1}{n+1} + \frac{1}{n+2}$

21. $\int_0^{\pi} [\cot x]\,dx$, [.] denotes the greatest integer function, is equal to **[AIEEE 2009]**

(a) $\frac{\pi}{2}$ (b) 1 (c) –1 (d) $\frac{-\pi}{2}$

22. The value of $\int_{-1}^{1}\left(e^{x^3} + e^{-x^3}\right)(e^x - e^{-x})\,dx$ is equal to

(a) $\frac{e^2}{2} - 2e$ (b) $e^2 - 2e$ (c) $2(e^2 - e)$ (d) 0

23. The value of $\lim_{n\to\infty}\left[\frac{n}{n^2+1^2} + \frac{n}{n^2+2^2} + \ldots + \frac{n}{n^2+n}\right]$ is

(a) $\frac{\pi}{4}$ (b) $\log 2$ (c) 0 (d) 1

24. Let $F(x) = f(x) + f\left(\frac{1}{x}\right)$ where $f(x) = \int_1^x \frac{\log t}{1+t}\,dt$. Then, $F(x)$ equals

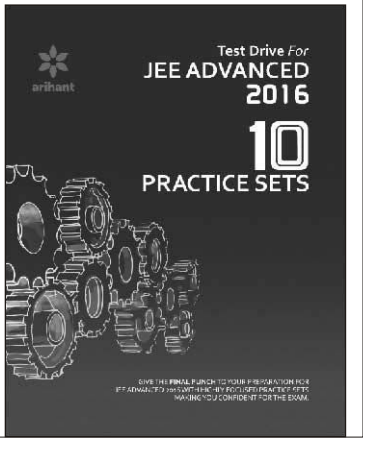
(a) $\frac{1}{2}$ (b) 0 (c) 1 (d) 2

25. The value of $\int_0^1 \tan^{-1}\left(\frac{2x-1}{1+x-x^2}\right)dx$ is

(a) 1 (b) 0 (c) –1 (d) None

JEE ADVANCED 2016

with Arihant's Specialised Practice Sets



ADVANCED PROPERTIES OF DEFINITE INTEGRAL, BETA AND GAMMA FUNCTIONS

Advanced Properties of Definite Integral

(i) If $f(x)$ is periodic function with period T {i.e. $f(x + T) = f(x)$}. Then, $\int_a^{a+T} f(x)\,dx$ is independent of 'a'

(a) $\int_0^{nT} f(x)\,dx = n\int_0^{T} f(x)\,dx,\ n \in I$

(b) $\int_{a+mT}^{a+nT} f(x)\,dx = (n-m)\int_0^{T} f(x)\,dx;\ m, n \in I$

(c) $\int_{a+mT}^{b+mT} f(x)\,dx = \int_a^b f(x)\,dx,\ m \in I$

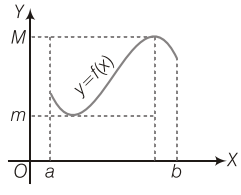
(d) $\int_{nT}^{a+nT} f(x)\,dx = \int_0^a f(x)\,dx$

(ii) If $f(x) \le \phi(x)$ for $x \in [a, b]$, then $\int_a^b f(x)\,dx \le \int_a^b \phi(x)\,dx$

(iii) If at every point x of an interval $[a, b]$ the inequalities $g(x) \le f(x) \le h(x)$ are full field, then $\int_a^b g(x)\,dx \le \int_a^b f(x)\,dx \le \int_a^b h(x)\,dx$

(iv) $\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx$

(v) If m is the least value and M is the greatest value of the function $f(x)$ in the interval $[a, b]$, then $m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$



(vi) If f is continuous on $[a, b]$, then there exist a number c in $[a, b]$ at which $f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx$ is called the mean value of the function $f(x)$ on the interval $[a, b]$.

(vii) If $f^2(x)$ and $g^2(x)$ are integrable on $[a, b]$, then $\left|\int_a^b f(x)\,g(x)\,dx\right| \le \left(\int_a^b f^2(x)\,dx\right)^{\frac{1}{2}}\left(\int_a^b g^2(x)\,dx\right)^{\frac{1}{2}}$

(viii) Let a function $f(x, \alpha)$ be continuous for $a \le x \le b$ and $c \le \alpha \le d$. Then, for any $\alpha \in [c, d]$, if $I(\alpha) = \int_a^b f(x, \alpha)\,dx$, then

$$\frac{dI(\alpha)}{d\alpha} = \int_a^b \frac{\partial f(x, \alpha)}{\partial \alpha}\,dx$$

(ix) If $f(t)$ is an odd function, then $\phi(x) = \int_a^x f(t)\,dt$ is an even function.

(x) If $f(t)$ is an even function, then $\phi(x) = \int_0^x f(t)\,dt$ is an odd function.

(xi) If $f(t)$ is an even function, then for non-zero 'a', $\int_a^x f(t)\,dt$ is not necessarily an odd function. If it will be an odd function, $\int_0^a f(t)\,dt = 0$

(xii) If $f(x)$ is continuous on $[a, \infty)$, then $\int_a^\infty f(x)\,dx$ is called an improper integral and is defined as $\int_a^\infty f(x)\,dx = \lim_{b \to \infty}\int_a^b f(x)\,dx$.

Leibnitz Rule for the Differentiation under Integral Sign

If the function $\phi(x)$ and $\psi(x)$ are defined on $[a, b]$ and are differentiable at a point $x \in (a, b)$ and $f(t)$ is continuous on $[\phi(a), \phi(b)]$, then $\frac{d}{dx}\left[\int_{\phi(x)}^{\psi(x)} f(t)\,dt\right] = \frac{d}{dx}\{\psi(x)\}\,f\{\psi(x)\} - \frac{d}{dx}\{\phi(x)\}\,f\{\phi(x)\}$

Walli's Formula

(i) $\int_0^{\pi/2} \sin^n x\,dx = \int_0^{\pi/2} \cos^n x\,dx$

$$= \begin{cases} \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots\frac{2}{3}, & \text{when } n \text{ is odd.} \\ \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}, & \text{when } n \text{ is even.} \end{cases}$$

(ii) $\int_0^{\pi/2} \sin^m x \cos^n x\,dx = \int_0^{\pi/2} \sin^n x \cos^m x\,dx.$

$$= \begin{cases} \frac{(m-1)(m-3)\ldots(1 \text{ or } 2).(n-1)(n-3)\ldots(1 \text{ or } 2)}{(m+n)(m+n-2)\ldots(1 \text{ or } 2)}\cdot\frac{\pi}{2}, & \text{when both } m, n \text{ are even integers} \\ \frac{(m-1)(m-3)\ldots(1 \text{ or } 2).(n-1)(n-3)\ldots(1 \text{ or } 2)}{(m+n)(m+n-2)\ldots(1 \text{ or } 2)}, & \text{when either } m \text{ or } n \text{ is odd integer} \end{cases}$$

Gamma Function

The improper integral $\int_0^\infty e^{-x}x^{n-1}dx,\ n \in I^+$ is defined as gamma function and is denoted by $\Gamma(n)$, i.e. $\Gamma(n) = \int_0^\infty e^{-x}x^{n-1}\,dx$, where $n \in Q^+$

Properties of Gamma Function

(i) $\Gamma(n+1) = n\,\Gamma n = n!$ (ii) $\Gamma 1 = 1$ (iii) $\Gamma 0 = \infty$, (iv) $\left|\frac{1}{2}\right| = \sqrt{\pi}$

(v) $\int_0^{\pi/2} \sin^m x \cos^n x\,dx = \frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{2\Gamma\left(\frac{m+n+2}{2}\right)}$

SPEED BOOSTER

(i) $\Gamma n\,\Gamma 1-n = \frac{\pi}{\sin n\pi},\ 0 < n < 1$ (ii) $\Gamma m\left|m+\frac{1}{2}\right. = \frac{\sqrt{\pi}}{2^{2m-1}}\Gamma 2m$

(iii) $\left|\frac{1}{n}\right|\left|\frac{2}{n}\right|\ldots\left|\frac{n-1}{n}\right. = \frac{(2\pi)^{\frac{n-1}{2}}}{n^{1/2}}$

Beta Function

The beta function is defined as

$B(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$, where $m, n > 0$

Properties of Beta Function

(i) $B(m, n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx$

(ii) $B(m, n) = \frac{\lfloor m \rfloor \lfloor n}{\lfloor m+n}$, where $m, n > 0$

(iii) $B(m, n) = B(n, m)$, where $m, n > 0$

MATHS MAXIMA

(i) $\sum_{r=1}^{n} r = \frac{n(n+1)}{2}$ (ii) $\sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$

(iii) $\sum_{r=1}^{n} r^3 = \left|\frac{n(n+1)}{2}\right|^2$

(iv) $\sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + \ldots + \sin[\alpha+(n-1)\beta]$

$= \frac{\sin\frac{n\beta}{2}\sin\left[\alpha+(n-1)\frac{\beta}{2}\right]}{\sin\frac{\beta}{2}}$

(v) $\cos\alpha + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + \ldots + \cos[\alpha+(n-1)\beta]$

$= \frac{\sin n\frac{\beta}{2}\cos\left[\alpha+(n-1)\frac{\beta}{2}\right]}{\sin\frac{\beta}{2}}$

(vi) $1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \frac{1}{6^2} + \ldots\infty = \frac{\pi^2}{12}$

(vii) $1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \ldots\infty = \frac{\pi^2}{6}$

(viii) $1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\infty = \frac{\pi^2}{8}$

(ix) $\frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \ldots\infty = \frac{\pi^2}{24}$

(x) $\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$ and $\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$

(xi) $\cosh\theta = \frac{e^{\theta} + e^{-\theta}}{2}$, $\sinh\theta = \frac{e^{\theta} - e^{-\theta}}{2}$

Some Important Results

(i) (a) $\int_0^a \frac{f(x)}{f(x)+f(a-x)}\,dx = \frac{a}{2}$

(b) $\int_a^b \frac{f(x)}{f(x)+f(a+b-x)}\,dx = \frac{b-a}{2}$

(ii) (a) $\int_0^{\pi/2} \frac{dx}{a+b\cos x} = \frac{2}{\sqrt{a^2-b^2}}\tan^{-1}\left(\sqrt{\frac{a+b}{a-b}}\right)$, $a > b > 0$

or

$\int_0^{\pi/2} \frac{dx}{a+b\cos x} = \frac{1}{\sqrt{b^2-a^2}}\log\left|\frac{\sqrt{b+a}-\sqrt{b-a}}{\sqrt{b+a}+\sqrt{b-a}}\right|$,

$0 < a < b$

(b) $\int_0^{\pi/2} \frac{dx}{a+b\sin x} = \frac{2}{\sqrt{a^2-b^2}}\tan^{-1}\left(\sqrt{\frac{a-b}{a+b}}\right)$, $a > b > 0$

or $\int_0^{\pi/2} \frac{dx}{a+b\sin x} = \frac{1}{\sqrt{b^2-a^2}}\log\left|\frac{\sqrt{b+a}+\sqrt{b-a}}{\sqrt{b+a}-\sqrt{b-a}}\right|$,

$0 < a < b$

(iii) (a) $\int_0^{\pi/2} \frac{\sin^n x}{\sin^n x + \cos^n x}\,dx = \frac{\pi}{4} = \int_0^{\pi/2} \frac{\cos^n x}{\sin^n x + \cos^n x}\,dx$

(b) $\int_0^{\pi/2} \frac{\tan^n x}{1+\tan^n x}\,dx = \frac{\pi}{4} = \int_0^{\pi/2} \frac{dx}{1+\tan^n x}$

(c) $\int_0^{\pi/2} \frac{dx}{1+\cot^n x} = \frac{\pi}{4} = \int_0^{\pi/2} \frac{\cot^n x}{1+\cot^n x}\,dx$

(d) $\int_0^{\pi/2} \frac{\tan^n x}{\tan^n x + \cot^n x}\,dx = \frac{\pi}{4} = \int_0^{\pi/2} \frac{\cot^n x}{\tan^n x + \cot^n x}\,dx$

(e) $\int_0^{\pi/2} \frac{\sec^n x}{\sec^n x + \operatorname{cosec}^n x}\,dx$

$= \frac{\pi}{4} = \int_0^{\pi/2} \frac{\operatorname{cosec}^n x}{\sec^n x + \operatorname{cosec}^n x}\,dx$ where, $n \in R$

(iv) $\int_0^{\pi/2} \frac{a^{\sin^n x}}{a^{\sin^n x} + a^{\cos^n x}}\,dx$

$= \int_0^{\pi/2} \frac{a^{\cos^n x}}{a^{\sin^n x} + a^{\cos^n x}}\,dx = \frac{\pi}{4}$

(v) (a) $\int_0^{\pi/2} \log\sin x\,dx = \int_0^{\pi/2} \log\cos x\,dx = -\frac{\pi}{2}\log 2$

(b) $\int_0^{\pi/2} \log\tan x\,dx = \int_0^{\pi/2} \log\cot x\,dx = 0$

(c) $\int_0^{\pi/2} \log\sec x\,dx$

$= \int_0^{\pi/2} \log\operatorname{cosec} x\,dx = \frac{\pi}{2}\log 2$

(vi) (a) $\int_0^\infty e^{-ax}\sin bx\,dx = \frac{b}{a^2+b^2}$

(b) $\int_0^\infty e^{-ax}\cos bx\,dx = \frac{a}{a^2+b^2}$

(c) $\int_0^\infty e^{-ax}x^n dx = \frac{n!}{a^n+1}$

Best Practice SHOTS

26. The value of $\int_0^{4\pi} |\cos x|\, dx$ is

(a) 4 (b) 7
(c) 8 (d) $\frac{1}{4}$

27. The value of $\int_{-5}^{5} f(x)\, dx$, where $f(x)$ = minimum $\{x+1\}, \{x-1\}, \forall\ x \in R$ and $\{.\}$ denotes fractional part of x, is equal to

(a) 3 (b) 4
(c) 5 (d) 6

28. The value of $\frac{d}{dx}\left(\int_{1/x}^{\sqrt{x}} \cos t^2\, dt\right)$ is equal to

(a) $\frac{1}{2\sqrt{x}}\cos x$ (b) $\frac{1}{2\sqrt{x}}\cos x + \frac{1}{x^2}\cos\left(\frac{1}{x^2}\right)$
(c) $\frac{1}{3\sqrt{x}}\cos x$ (d) None of these

29. If $\frac{d}{dx}\left(\int_0^y e^{-t^2}\, dt + \int_0^{x^2} \sin^2 t\ dt\right) = 0$, then $\frac{dy}{dx}$ is equal to

(a) $-2xe^{y^2}\sin^2 x^3$ (b) $-2xe^{y^2}$
(c) $-2xe^{y^2}\sin^2 x^2$ (d) 1

30. If $\int_{\pi/3}^{x} \sqrt{3-\sin^2 t}\ dt + \int_0^y \cot t\ dt = 0$, then the value of $\frac{dy}{dx}$ is equal to

(a) $\frac{\sqrt{3+\sin^2 x}}{\cos y}$ (b) $\frac{\sqrt{2-\sin^2 x}}{\sin y}$
(c) $\frac{-\sqrt{3-\sin^2 x}}{\cos y}$ (d) $\frac{\sqrt{3-\sin^2 x}}{\cos y}$

31. Let f be a non-negative function defined on the interval $[0, 1]$. If $\int_0^x \sqrt{1-(f'(t))^2}\ dt = \int_0^x f(t)\, dt, 0 \le x \le 1$ and $f(0) = 0$, then **[IIT JEE 2009]**

(a) $f\left(\frac{1}{2}\right) < \frac{1}{2}$ and $f\left(\frac{1}{3}\right) > \frac{1}{3}$ (b) $f\left(\frac{1}{2}\right) > \frac{1}{2}$ and $f\left(\frac{1}{3}\right) > \frac{1}{3}$
(c) $f\left(\frac{1}{2}\right) < \frac{1}{2}$ and $f\left(\frac{1}{3}\right) < \frac{1}{3}$ (d) $f\left(\frac{1}{2}\right) > \frac{1}{2}$ and $f\left(\frac{1}{3}\right) < \frac{1}{3}$

32. $\int_0^{\pi} x\sin^4 x\, dx$ is equal to **[IIT JEE 2005]**

(a) $\frac{3\pi}{16}$ (b) $\frac{3\pi^2}{16}$ (c) $\frac{16\pi}{3}$ (d) $\frac{16\pi^2}{3}$

33. If $\int_0^{\pi/2} \sin^6 x\, dx = \frac{5\pi}{32}$, then the value of $\int_{-\pi}^{\pi} (\sin^6 x + \cos^6 x)\, dx$ is

(a) $\frac{5\pi}{8}$ (b) $\frac{5\pi}{16}$
(c) $\frac{5\pi}{2}$ (d) $\frac{5\pi}{4}$

34. The value of $\int_0^{\pi/2} \sin^8 x\, dx$ is

(a) $\frac{105\pi}{32\ (4!)}$ (b) $\frac{105\pi}{14(4!)}$
(c) $\frac{105}{16\pi\ (4!)}$ (d) None of these

35. The value of $\int_0^1 \sqrt{(1+x)\ (1+x^3)}\ dx$ is

(a) less than or equal to $\sqrt{\frac{15}{8}}$
(b) greater than or equal to $\sqrt{\frac{15}{8}}$
(c) greater than $\sqrt{\frac{15}{8}}$
(d) None of the above

36. The value of $\int_0^1 \frac{dx}{\sqrt{4-x^2-x^3}}$ lies in the interval

(a) $\left[\frac{\pi}{6}, \frac{\pi}{4\sqrt{2}}\right]$ (b) $\left(\frac{\pi}{6}, \frac{\pi}{4\sqrt{2}}\right)$
(c) $\left[\frac{\pi}{6}, \frac{\pi}{4\sqrt{2}}\right)$ (d) $\left(\frac{\pi}{6}, \frac{\pi}{4\sqrt{2}}\right]$

37. The minimum odd value of a $(a > 1)$ for which $\int_{10}^{19} \frac{\sin x}{1+x^a} dx < \frac{1}{9}$, is equal to

(a) 1 (b) 3
(c) 5 (d) 9

38. The value of $\int_0^{25} e^{x-[x]} dx$ is equal to (where, $[\cdot]$ denotes the greatest integer function)

(a) 25 $(e+1)$ (b) 25 $(e-1)$
(c) $-25\ (e+1)$ (d) None of these

39. The value of $\int_0^{2n\pi} [\sin x + \cos x]\, dx$ is equal to (where, $[.]$ denotes the greatest integer function)

(a) $-n\pi$ (b) $n\pi$
(c) $-2n\pi$ (d) None of these

40. The value of x satisfying $\int_0^{2[x+14]} \left\{\frac{x}{2}\right\} dx = \int_0^{\{x\}} [x+14]\, dx$, is equal to (where $[.]$ and $\{.\}$ denote the greatest integer and fractional part of x respectively)

(a) $[-14, -13)$ (b) $(0, 1)$
(c) $(-15, -14]$ (d) None of these

41. If $y = \int_0^x f(t)\sin\{k(x-t)\}\, dt$, then $\frac{d^2y}{dx^2} + k^2 y$ is equal to

(a) $k2f(x)$ (b) $-k2f(x)$
(c) $kf(x)$ (d) $f(x)$

{JEE FINAL TOUCH}

42. The value of $\int_0^{\pi/2} \sin^5 x\, dx$ is

(a) $\frac{4}{15}$ (b) $\frac{8}{5}$ (c) $\frac{8}{15}$ (d) $\frac{4}{5}$

43. If $\int_0^{x^2} f(t)\, dt = x\cos\pi x$, then the value of $f(4)$ is

(a) 1 (b) $\frac{1}{4}$ (c) -1 (d) $\frac{-1}{4}$

44. If $\int_{\sin x}^{1} t^2 f(t)\, dt = 1 - \sin x, \forall\, x \in \left[0, \frac{\pi}{2}\right]$, then $f\left(\frac{1}{\sqrt{3}}\right)$ is

(a) 3 (b) $\sqrt{3}$ (c) $\frac{1}{3}$ (d) None

45. If $f(x)$ is differentiable and $\int_0^{t^2} xf(x)\, dx = \frac{2}{5}t^5$, then $f\left(\frac{4}{25}\right)$ equals

(a) $\frac{2}{5}$ (b) $\frac{-5}{2}$ (c) 1 (d) $\frac{5}{2}$

46. $\int_0^1 \log\left\{\sin\left(\frac{\pi x}{2}\right)\right\} dx$ is equal to

(a) $\frac{-\pi}{2}\log 2$ (b) $-\log 2$ (c) $\frac{-2}{\pi}\log 2$ (d) $\frac{\pi}{2}\log 2$

47. The value of $\left[\int_0^{\sin^2\theta} \sin^{-1}\sqrt{\phi}\, d\phi + \int_0^{\cos^2\theta} \cos^{-1}\sqrt{\phi}\, d\phi\right]$ is

(a) π (b) $\frac{\pi}{2}$ (c) $\frac{\pi}{3}$ (d) $\frac{\pi}{4}$

48. The value of $\frac{dx}{(a^2\sin^2 x + b^2\cos^2 x)^2}$ is equal to

(a) $\frac{\pi(a^2+b^2)}{4a^3b^3}$ (b) $\frac{2\pi}{4a^3b^3}$

(c) $\frac{(a^2+b^2)}{4a^3b^3}$ (d) $3\pi(a^2+b^2)$

49. The value of $\int_{-4}^{-5} e^{(x+5)^2} dx + 3\int_{1/3}^{2/3} e^{9\left(x-\frac{2}{3}\right)^2} dx$ is

(a) 1 (b) 2 (c) 3 (d) 0

50. The value of $\int_1^3 \sqrt{3+x^3}\, dx$ lies in the interval

(a) $[4, 2\sqrt{3})$ (b) $[4, 2\sqrt{3}]$
(c) $(4, 2\sqrt{3}]$ (d) $(4, 2\sqrt{3})$

51. The value of $\int_0^\infty e^{-x}x^3\, dx$ is equal to

(a) 4 (b) 6 (c) 5 (d) 9

52. The value of $\int_0^{\pi/2} \sin^4 x \cdot \cos^6 x\, dx$ is

(a) $\frac{3\pi}{412}$ (b) $\frac{4\pi}{906}$ (c) $\frac{3\pi}{512}$ (d) $\frac{6\pi}{709}$

53. The value of $\int_0^\infty e^{-a^2x^2} dx$ is equal to

(a) $\frac{\sqrt{\pi}}{2a}$ (b) $\frac{\pi}{2a}$

(c) $\frac{\pi}{\sqrt{2a}}$ (d) None of these

Answers

1. (a)	**2.** (a)	**3.** (a)	**4.** (c)	**5.** (b)	**6.** (d)	**7.** (a)	**8.** (b)	**9.** (b)	**10.** (b)
11. (a)	**12.** (a)	**13.** (c)	**14.** (b)	**15.** (b)	**16.** (a)	**17.** (b)	**18.** (a)	**19.** (b)	**20.** (c)
21. (d)	**22.** (d)	**23.** (a)	**24.** (a)	**25.** (b)	**26.** (c)	**27.** (c)	**28.** (b)	**29.** (c)	**30.** (c)
31. (c)	**32.** (b)	**33.** (d)	**34.** (a)	**35.** (a)	**36.** (a)	**37.** (b)	**38.** (b)	**39.** (a)	**40.** (a)
41. (c)	**42.** (c)	**43.** (b)	**44.** (a)	**45.** (a)	**46.** (b)	**47.** (d)	**48.** (a)	**49.** (d)	**50.** (b)
51. (b)	**52.** (c)	**53.** (a)							

Notice Board

JKCET will be on May 21 & 22, 2016

The Jammu & Kashmir Board of Professional Examination has announced the dates for Jammu & Kashmir Common Entrance Test (JKCET) 2016, conducted for admissions to undergraduate engineering courses across engineering colleges in Jammu & Kashmir. JKCET 2016 is scheduled to be conducted on May 21 and May 22, 2016. Apart from the exam dates, the notification released by the exam conducting authority also published the documents required to be kept ready by a student before filling the Application Form of JKCET 2016. Candidates aspiring for the exam have to be ready with the below mentioned documents for filling the application form:

- Permanent Resident Certificate (State Subject)
- Date of Birth Certificate
- Reserved Category Certificate (if applicable)
- Marks Certificate/Marks Card of qualifying Examination. (If qualifying exam is passed)
- Tuition Fee Waiver Certificates (where Annual income of the parents is less than Rs. 4.50/- Lacs)

Formula at a Glance

1. If $F(x)$ is the anti-derivative of $f(x)$, then

$$\int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a)$$

this is called fundamental theorem of integral calculus.

2. **Properties of Definite Integral**

(i) $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$

(ii) $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

(iii) $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)dx$, $a < c < b$ {this property is also true when c does not lie between a and b}

(iv) $\int_a^b f(x)\,dx = \int_a^b f(a + b - x)\,dx$

(v) $\int_0^{2a} f(x)\,dx = \int_0^a \{f(x) + f(2a - x)\}\,dx$

$$\Rightarrow \int_0^{2a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx, & \text{if } f(2a - x) = f(x) \\ 0, & \text{if } f(2a - x) = -f(x) \end{cases}$$

(vi) $$\int_{-a}^a f(x)\,dx = \begin{cases} 0, & \text{if } f(x) \text{ is an odd function} \\ 2\int_0^a f(x)\,dx, & \text{if } f(x) \text{ is an even function} \end{cases}$$

(vii) If f is periodic function in properties VII to XI with period T, then $\int_0^{nT} f(x)\,dx = n\int_0^T f(x)\,dx$, $n \in N$

(viii) $\int_a^{a+T} f(x)\,dx$, $(a > 0)$ is independent of 'a'

$\therefore$ We can write $\int_a^{a+T} f(x)\,dx = \int_0^T f(x)\,dx$

(ix) $\int_a^{a+nT} f(x)\,dx = n\int_0^T f(x)\,dx$, $n \in N$

(x) $\int_{mT}^{nT} f(x)\,dx = (n - m)\int_0^T f(x)\,dx$, $m, n \in N$

(a) $\int_{a+mT}^{a+nT} f(x)\,dx = \int_{mT}^{nT} f(x)\,dx$

$= (n - m)\int_0^T f(x)\,dx$, $m,n \in N$

(b) $\int_{a+mT}^{b+mT} f(x)\,dx = \int_a^b f(x + mT)\,dx$

$= \int_a^b f(x)\,dx$

(xi) $\int_a^b f(x)\,dx = (b - a)\int_0^1 f\{(b - a)\,x + a\}\,dx$

(xii) $\frac{d}{dx}\int_a^b f(x,t)\,dt = \int_a^b \frac{\partial}{\partial x}\{f(x,t)\}dt$, where $\frac{\partial}{\partial x}\{f(x,t)\}$ is derivative of f w.r.t. 'x' keeping 't' as constant.

(xiii) $\frac{d}{dx}\int_{\phi(x)}^{\psi(x)} f(x,t)\,dt = f\{x, \psi(x)\}\cdot\psi'(x)$

$- f\{x, \phi(x)\}\cdot\phi'(x) + \int_{\phi(x)}^{\psi(x)} \frac{\partial}{\partial x} f(x,t)\,dt$

(xiv) If $f(x) \le \phi(x)$ for $x \in [a, b]$, then $\int_a^b f(x)\,dx \le \int_a^b \phi(x)\,dx$

(xv) If at every point x of an interval $[a, b]$ the inequalities $g(x) \le f(x) \le h(x)$ are full field, then

$\int_a^b g(x)\,dx \le \int_a^b f(x)\,dx \le \int_a^b h(x)\,dx$

(xvi) If f is continuous on $[a, b]$, then there exist a number c in $[a, b]$ of which $f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx$ is called the mean value of the function $f(x)$ on the interval $[a, b]$.

(xvii) If $f^2(x)$ and $g^2(x)$ are integrable on $[a, b]$, then

$$\left|\int_a^b f(x)\,g(x)\,dx\right| \le \left(\int_a^b f^2(x)\,dx\right)^{\frac{1}{2}}\left(\int_a^b g^2(x)\,dx\right)^{\frac{1}{2}}$$

(xviii) Let a function $f(x, \alpha)$ be continuous for $a \le x \le b$ and $c \le \alpha \le d$. Then, for any $\alpha \in [c, d]$, if $I(\alpha) = \int_a^b f(x, \alpha)\,dx$, then $\frac{dI(\alpha)}{d\alpha} = \int_a^b \frac{\partial f(x,\alpha)}{\partial \alpha}\,dx$

(xix) $\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx$

(xx) If m is the least value and M is the greatest value of the function $f(x)$ on the interval $[a, b]$, then $m(b - a) \le \int_a^b f(x)\,dx \le M(b - a)$

3. $\frac{d}{dx}\int_{\phi(x)}^{\psi(x)} f(t)\,dt = \frac{d}{dx}\{\psi(x)\}\cdot f\{\psi(x)\} - \frac{d}{dx}\{\phi(x)\}\cdot f\{\phi(x)\}$ this is called Newton - Leibnitz rule.

4. Walli's Formula

(i) $\int_0^{\pi/2} \sin^n x\,dx = \int_0^{\pi/2} \cos^n x\,dx$, $n \in N$

$$= \begin{cases} \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdots\frac{2}{3}\cdot 1, & \text{when } n \text{ is odd} \\ \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdots\frac{1}{2}\cdot\frac{\pi}{2}, & \text{when } n \text{ is even} \end{cases}$$

(ii) $\int_0^{\pi/2} \sin^m x\cos^n x\,dx = \int_0^{\pi/2} \sin^n x\cos^n x\,dx$, $m, n \in I^+$

$$= \begin{cases} \frac{(m-1)(m-3)\ldots(1 \text{ or } 2).}{(m+n)(m+n-2)\ldots(1 \text{ or } 2)}\cdot\frac{\pi}{2}, & \text{when both } m \text{ and } n \text{ are even integers} \\ \frac{(m-1)(m-3)\ldots(1 \text{ or } 2).}{(m+n)(m+n-2)\ldots(1 \text{ or } 2)}, & \text{when either } m \text{ or } n \text{ is odd integer} \end{cases}$$

5. Gamma function $\overline{|(n)} = \int_0^\infty e^{-x}x^{n-1}dx$, $n \in$ is known as gamma function

(i) $\overline{|(n+1)} = n\overline{|n} = n!$

(ii) $\overline{|(1)} = 1$

(iii) $\overline{|0} = \infty$

(iv) $\overline{\left|\left(\frac{1}{2}\right)} = \sqrt{\pi}$

6. **Beta Function** $B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$, where $m,n > 0$

(i) $B(m, n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}dx$

(ii) $B(m, n) = \frac{\overline{|m}\,\overline{|n}}{\overline{|m+n}}$, where $m, n > 0$

(iii) $B(m, n) = B(n, m)$, where $m, n > 0$

Application of Derivative

DERIVATIVE AS RATE MEASURE, ERRORS AND APPROXIMATIONS

Derivative as Rate Measure

If a variable quantity y is some function of 't' (time), i.e. $y = f(t)$, then small change in time Δt causes a corresponding change Δy in y.

Then, average rate of change $= \frac{\Delta y}{\Delta t}$

When limit $\Delta t \to 0$ applied, the rate of change becomes instantaneous.

i.e. $\lim\limits_{\Delta t \to 0} \frac{\Delta y}{\Delta t}$ = instantaneous rate of change of y w.r.t. 't'.

Hence, it is clear that the rate of change of any variable with respect to some other variable is the derivative of first variable with respect to other variable.

In order to solve problems related to rate measure we use following steps

***Step* I** Assign a variable to each quantity. Draw a diagram, if needed.

***Step* II** Write given values of variable and their rate of changes with respect to 't'.

***Step* III** Find an equation giving the relationship between the variable.

***Step* IV** Differentiate both sides of this equation with respect to 't'.

***Step* V** Replace the variable and their derivatives by the numerical values found in step II and solve.

Velocity and Acceleration in Rectilinear and Circular Motion

Rectilinear Motion

Students have already studied velocity and acceleration in depth. Here, we recall only definition of velocity as well as acceleration in terms of derivative.

The velocity of a moving particle is defined as the rate of change of its displacement with respect to time and the acceleration is defined as the rate of change its velocity with respect to time. If the displacement 's' of a particle moving in a straight line at a time 't' is given by $s = f(t)$, then velocity at time 't', $(v) = \frac{ds}{dt}$

Acceleration at time 't' $(a) = \frac{dv}{dt} = \frac{d^2s}{dt^2} = v \cdot \frac{dv}{ds}$

Note

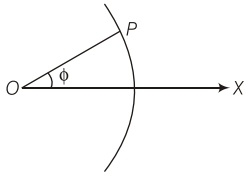
(i) When a particle moving in a straight line comes to rest, then

$$\frac{ds}{dt} = 0 \text{ and } \frac{d^2s}{dt^2} = 0$$

(ii) When a particle moving in a straight line is instantaneously at rest, then $\frac{ds}{dt} = 0, \frac{d^2s}{dt^2} \neq 0$

Circular Motion

Let OX be the initial line and let P be the position of a particle moving on a curve at any time 't' such that $\angle XOP = \phi$, then



Angular velocity $= \frac{d\phi}{dt} = \omega$

Angular acceleration $= \frac{d^2\phi}{dt^2} = \frac{d\omega}{dt}$

Approximations

Let $y = f(x)$ be a given function defined by $f : X \to R$, where $X \subset R$

Let for small increment Δx in independent variable x corresponding increment Δy in dependent variable y. Then,

$$y = f(x) \text{ a nd } \quad y + \Delta y = f(x + \Delta x)$$

$$\Rightarrow \qquad \Delta y = f(x + \Delta x) - f(x)$$

$$\therefore \qquad \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \frac{dy}{dx}$$

Self Study Guide for Online
BITSAT 2016

Best Practice SHOTS

1. If the percentage error in measuring the surface area of a sphere is $\alpha\%$, then the error in its volume is

(a) $\frac{3}{2}\alpha\%$ (b) $\frac{2}{3}\alpha\%$ (c) $3\alpha\%$ (d) None

2. The radius of the base of a cone is increasing at the rate of 3 cm/min and the altitude is decreasing at the rate of 4 cm/min. The rate of change of lateral surface when the radius 7 cm and altitude 24 cm is

(a) 54π cm^2/min (b) 7π cm^2/min

(c) 27 cm^2/min (d) None of these

3. If errors of 1% each is made in the base radius and height of a cylinder, then the percentage error in its volume is

(a) 1% (b) 2% (c) 3% (d) None

4. A variable triangle is inscribed in a circle of radius R. If the rate of change of a side is R times the rate of change of the opposite angle, then the angle is

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{3}$ (d) $\frac{\pi}{2}$

5. A point on the parabola $y^2 = 18x$ at which the ordinate increases at twice the rate of the abscissa is

(a) $\left(\frac{9}{8}, \frac{9}{2}\right)$ (b) $(2, -4)$ (c) $\left(-\frac{9}{8}, \frac{9}{2}\right)$ (d) $(2, 4)$

6. A particle's velocity v at time t is given by $v = 2e^{2t}\cos\frac{\pi t}{3}$.

The least value of t at which the acceleration becomes zero, is

(a) 0 (b) $\frac{3}{2}$

(c) $\frac{3}{\pi}\tan^{-1}\left(\frac{6}{\pi}\right)$ (d) $\frac{3}{\pi}\cot^{-1}\left(\frac{6}{\pi}\right)$

7. The semi-vertical angle of a cone is 45°. If the height of the cone is 20.025, then its approximate lateral surface area is

(a) $401\sqrt{2}\pi$ (b) $400\sqrt{2}\pi$

(c) $399\sqrt{2}\pi$ (d) None of these

8. For what values of x is the rate of increase of $x^3 - 5x^2 + 5x + 8$ is twice the rate of increase of x?

(a) $-3, -\frac{1}{3}$ (b) $-3, \frac{1}{3}$ (c) $3, -\frac{1}{3}$ (d) $3, \frac{1}{3}$

9. The area of a triangle is computed using the formula $S = 1/2\ bc\sin A$. If the relative errors made in measuring b, c and calculating S are respectively 0.02, 0.01 and 0.13, the approximate error in A when $A = \pi/6$, is

(a) 0.05 radians (b) 0.01 radians

(c) 0.05 degree (d) 0.01 degree

10. The weight w of a certain stock of fish is given by $W = nw$, where n is the size of stock and w is the average weight of a fish. If n and w change with time t as $n = 2t^2 + 3$ and $w = t^2 - t + 2$, then the rate of change of W with respect to t at $t = 1$ is **[AIEEE 2012]**

(a) 1 (b) 13 (c) 5 (d) 8

INCREASING AND DECREASING FUNCTIONS (MONOTONOCITY)

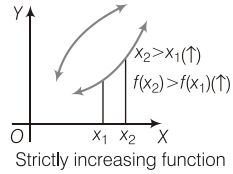
Monotonocity is the most important behaviour of the functions. The word monotonocity means increasing (ascending) or decreasing (descending). If function output is increasing on increasing input in its entire domain, then the function is said to be an increasing function and if output of the function decreases on increasing input, then function is called decreasing function.

Strictly Increasing Function

A function $f(x)$ is known as strictly increasing function in its domain, if $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$

or

If a function is strictly increasing in $[a, b]$, then $f'(x) > 0$, $\forall\ x \in [a, b]$



Strictly increasing function

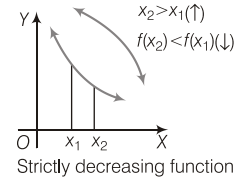
Strictly Decreasing Function

A function $f(x)$ is said to be strictly decreasing function in its domain, if

$$x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$$

or

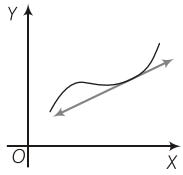
If the function is strictly decreasing in $[a, b]$, then $f'(x) < 0$ $\forall\ x \in [a, b]$



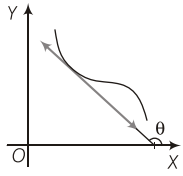
Strictly decreasing function

Geometrical Interpretation

(i) If $f(x)$ is strictly increasing in $[a, b]$, then tangent to curve has a positive slope at all points which lies between a and b, i.e. slope of tangent line will be acute.

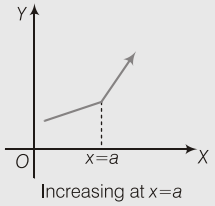


(ii) If $f(x)$ is strictly decreasing in $[a, b]$, then tangent to curve has negative slope at all points of the curve lying in between a and b, i.e. $f'(x) < 0, \forall\, x \in [a, b]$.

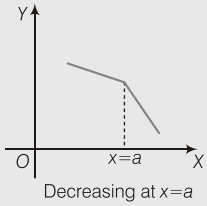


Note If $f'(x) \geq 0, \forall\, x \in [a, b]$, then $f(x)$ is simply called an increasing function in $[a, b]$. Similarly, if $f'(x) \leq 0, \forall\, x \in [a, b]$, then $f(x)$ is simply called a decreasing function in $[a, b]$.

We can take of monotonocity of $f(x)$ at $x = a$ only if $x = a$ lies in the domain of f, without any consideration of continuity or differentiability of $f(x)$ at $x = a$.

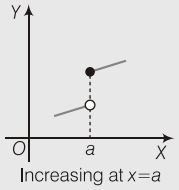


Increasing at $x=a$



Decreasing at $x=a$

(Not differentiable at $x=a$)



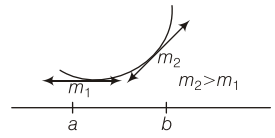
Increasing at $x=a$
(Discontinuous)

Increasing at $x=a$
(Not differentiable)

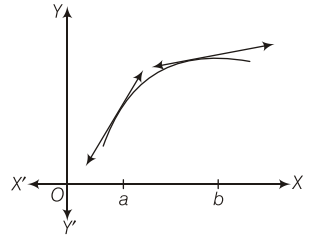
Concavity of Curves

A curve is called concave downward, if all points of the curve lie below the tangent to the curve at any point and curve is called concave upward, if all the points lie above the tangent drawn to the curve at any point. Geometrically, it can be understand in following ways.

(i) If the curve is concave upwards, its slope is increases as x increases, i.e. $f'(x)$ is strictly increasing in $[a, b]$.
$\Rightarrow f''(x) > 0, \forall\, x \in [a, b]$



(ii) If the curve is concave downwards, then the slope of the tangent decreases as x increases, i.e. $f'(x)$ is strictly decreasing in $[a, b]$.
$\Rightarrow f''(x) < 0, \forall\, x \in [a, b]$



Monotonocity in an Interval

Let I be an open interval contained in the domain of real-valued function f

(i) Function f is increases on I, if $x_1 < x_2$ in I
$\Rightarrow f(x_1) \leq f(x_2), \forall\, x_1$ and $x_2 \in I$

(ii) Function f is decreasing on I, if $x_1 < x_2$ in I
$\Rightarrow f(x_1) \geq f(x_2), \forall\, x_1$ and $x_2 \in I$

(iii) Function f is strictly increasing on I, if $x_1 < x_2$ in I
$\Rightarrow f(x_1) < f(x_2), \forall\, x_1 x_2 \in I$

Point of Inflection

A point on the curve, where the curve changes its nature of concavity is called point of inflection.

If f having derivative of all order at $x = a$, such that $f'(a) = 0 = f''(a)$ and $f'''(a) \neq 0$, then the point $x = a$ is known as point of inflection.

Note Monotonic function have atmost one root.

Properties of Monotonic Functions

(i) If $f(x)$ is strictly increasing function on an interval $[a, b]$, then f^{-1} exist and also a strictly increasing function.

(ii) If $f(x)$ is strictly increasing function on $[a, b]$ such that it is continuous, then f^{-1} is continuous on $[f(a), f(b)]$.

(iii) If $f(x)$ and $g(x)$ are strictly increasing (or decreasing) function on $[a, b]$, then $gof(x)$ is strictly increasing (or decreasing) function on $[a, b]$.

(iv) If one of the two functions $f(x)$ and $g(x)$ is strictly increasing and other is strictly decreasing, then $gof(x)$ is strictly decreasing on $[a, b]$.

(v) If $f(x)$ is continuous on $[a, b]$ such that $f'(c) > 0$ for each $c \in (a, b)$, then $f(x)$ is strictly increasing function on $[a, b]$.

(vi) If $f(x)$ is continuous on $[a, b]$ such that $f'(c) < 0$ for each $c \in (a, b)$, then $f(x)$ is strictly decreasing function on $[a, b]$.

Best Practice SHOTS

11. If $f(x) = \frac{x}{\sin x}$ and $g(x) = \frac{x}{\tan x}$, where $0 < x \le 1$, then in this interval

(a) both $f(x)$ and $g(x)$ are increasing functions
(b) both $f(x)$ and $g(x)$ are decreasing functions
(c) $f(x)$ is an increasing function
(d) $g(x)$ is an increasing function

12. If $f(x) = \begin{cases} 3x^2 + 12x - 1, & -1 \le x \le 2 \\ 37 - x, & 2 < x \le 3 \end{cases}$, then

(a) $f(x)$ is increasing on $[-1, 2]$ (b) $f(x)$ is continuous on $[-1, 3]$
(c) $f'(2)$ does not exist (d) All of these

13. If a, b and c are real, then $f(x) = \begin{vmatrix} x + a^2 & ab & ac \\ ab & x + b^2 & bc \\ ac & bc & x + c^2 \end{vmatrix}$ is decreasing on

(a) $\left\{-\frac{2}{3}(a^2 + b^2 + c^2), 0\right\}$ (b) $\left\{0, \frac{2}{3}(a^2 + b^2 + c^2)\right\}$
(c) $\left\{\left(\frac{a^2 + b^2 + c^2}{3}\right), 0\right\}$ (d) None of these

14. If $f'(x^2 - 4x + 3) > 0$ for all $x \in (2, 3)$, then $f(\sin x)$ is increasing on

(a) $\bigcup_{n \in Z}\left(2n\pi, (4n + 1)\frac{\pi}{2}\right)$ (b) $\bigcup_{n \in Z}\left((4n - 1)\frac{\pi}{2}, 2n\pi\right)$
(c) R (d) None of these

15. The length of the longest interval in which the function $3\sin x - 4\sin^3 x$ is increasing, is **[IIT 2002]**

(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{2}$ (c) $\frac{3\pi}{2}$ (d) π

16. Let $f(x) = \begin{cases} xe^{ax}, & x \le 0 \\ x + ax^2 - x^3, & x > 0 \end{cases}$, where a is a positive constant. Then, the interval in which $f'(x)$ is increasing, is

(a) $\left(0, \frac{a}{3}\right)$ (b) $\left(-\frac{2}{a}, 0\right)$ (c) $\left(-\frac{2}{a}, \frac{a}{3}\right)$ (d) None of these

17. If the function $f(x) = 3\cos|x| - 6ax + b$ increases for all $x \in R$, then the range of values of a is given by

(a) $\left(-\frac{1}{2}, \infty\right)$ (b) $\left(-\infty, \frac{-1}{2}\right)$ (c) $(-\infty, -2)$ (d) $(-2, \infty)$

18. For all $x \in (0, 1)$, **[IIT 2000]**

(a) $e^x < (1 + x)$ (b) $\log_e(1 + x) < x$
(c) $\sin x > x$ (d) $\log_e x > x$

19. For the function $f(x) = x\cos\frac{1}{x}$, $x \ge 1$, which one of the following is incorrect?

(a) For atleast one x in the interval $[1, \infty)$, $f(x + 2) - f(x) < 2$
(b) $\lim_{x \to \infty} f'(x) = 1$
(c) For all x in the interval $[1, \infty)$, $f(x + 2) - f(x) > 2$
(d) $f'(x)$ is directly decreasing in the interval $[1, \infty)$

20. Let $f(x) = (1 - x)^2\sin^2 x + x^2$ for all $x \in R$ and let $g(x) = \int_1^x \left\{\frac{2(t - 1)}{t + 1} - \log_e t\right\} f(t)\,dt$ for all $x \in (1, \infty)$. Then, which of the following is true? **[IIT 2012]**

(a) g is increasing on $(1, \infty)$
(b) g is decreasing on $(1, \infty)$
(c) g is increasing on $(1, 2)$ and decreasing on $(2, \infty)$
(d) g is decreasing on $(1, 2)$ and increasing on $(2, \infty)$

TANGENT AND NORMAL

Slope of Tangent

If a tangent is drawn to the curve $y = f(x)$ at a point (x_1, y_1) and this tangent makes an angle ϕ with positive direction of X-axis, then slope of tangent $= \tan\phi = \left(\frac{dy}{dx}\right)_{(x_1, y_1)}$

Note

(i) If the tangent at any point P on the curve is parallel to the X-axis, then $\frac{dy}{dx} = 0$ at the point P.

(ii) If the tangent at any point on the curve is parallel to the Y-axis, then $\frac{dy}{dx} = \infty$ or $\frac{dx}{dy} = 0$.

(iii) If the tangent is equally inclined to the axes, then $\frac{dy}{dx} = \pm 1$

(iv) If intercept made by the tangent are equal in positive coordinate axes, then $\frac{dy}{dx} = -1$

Equation of Tangent

The equation of tangent at $P(x_1, y_1)$ is

$$y - y_1 = \left(\frac{dy}{dx}\right)_{(x_1, y_1)} (x - x_1)$$

Note

(i) If tangent is drawn from external point (α, β), then $\frac{y_1 - \beta}{x_1 - \alpha} = \left(\frac{dy}{dx}\right)_{(x_1, y_1)}$, where (x_1, y_1) is point of contact.

(ii) A line $ax + by + c = 0$ is tangent to the curve, if $-\frac{a}{b} = \left(\frac{dy}{dx}\right)_{(x_1, y_1)}$, where (x_1, y_1) is point of contact.

Slope of Normal

The normal to a curve is a line at point $P(x_1, y_1)$ which is perpendicular to tangent at P.

$$\text{Slope of normal} = -\frac{1}{\text{Slope of tangent}} = -\frac{1}{\left(\frac{dy}{dx}\right)_{(x_1, y_1)}}$$

Note (i) If the normal is parallel to X-axis, then

$$-\left(\frac{dx}{dy}\right)_{(x_1, y_1)} = 0 \text{ or } \left(\frac{dx}{dy}\right)_{(x_1, y_1)} = 0$$

(ii) If the normal is perpendicular to X-axis (or parallel to Y-axis), then $-\left(\frac{dy}{dx}\right)_{(x_1, y_1)} = 0$ or $\left(\frac{dy}{dx}\right)_{(x_1, y_1)} = 0$

(iii) If normal is equally inclined to axes, then $\frac{dy}{dx} = \pm 1$

Equation of Tangent and Normal in Parametric Form

If $x = f(t)$ and $y = g(t)$ and $\left|\frac{dy}{dx}\right|_P = \frac{g'(t_0)}{f'(t_0)}$

here coordinates of point P are $x_0 = f(t_0)$, $y_0 = g(t_0)$

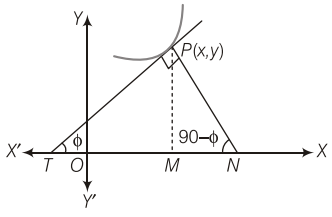
Now, the equation of tangent is $(y - g(t_0)) = \frac{g'(t_0)}{f'(t_0)}(x - f(t_0))$

and equation of normal is

$$(y - g(t_0)) = -\frac{f'(t_0)}{g'(t_0)}(x - f(t_0))$$

Length of Tangent, Normal, Subtangent and Subnormal

Let P be any point on the curve $y = f(x)$



(i) **Length of Tangent**

$$PT = |y \operatorname{cosec} \phi| = |y\sqrt{1 + \cot^2 \phi}| = \left|y\sqrt{1 + \left(\frac{dx}{dy}\right)^2}\right|$$

(ii) **Length of Normal**

$$PN = |y \operatorname{cosec} (90 - \phi)|$$

$$= |y \sec \phi| = |y\sqrt{1 + \tan^2 \phi}| = \left|y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\right|$$

(iii) **Length of Subtangent**

$$TM = |y \cot \phi| = \left|\frac{y}{\tan \phi}\right| = \left|y\left(\frac{dx}{dy}\right)\right|$$

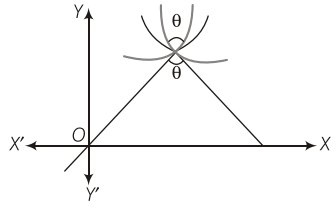
(iv) **Length of Subnormal**

$$MN = |y \cot (90 - \phi)| = |y \tan \phi| = \left|y \cdot \left(\frac{dy}{dx}\right)\right|$$

Angle of Intersection of Two Curves

The angle between two curves is basically angle between the tangents to the curves at the point of intersection of two curves. If m_1 and m_2 are slopes of tangent at the point of contact, then

$$\tan \theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$



Note

(i) If $\theta = \frac{\pi}{2}$, then $m_1 m_2 + 1 = 0 \Rightarrow m_1 \cdot m_2 = -1$

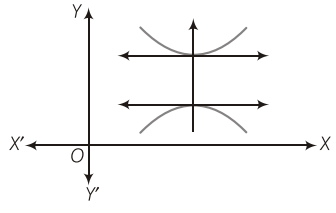
In this situation the curves are known as orthogonal curves.

(ii) If $\theta = 0$, then $m_1 = m_2$, then the curves are tangential.

(iii) Length of perpendicular from origin to the tangent is

$$\left|\frac{y_1 - x_1\left(\frac{dy}{dx}\right)_{(x_1, y_1)}}{\sqrt{1 + \left(\frac{dy}{dx}\right)^2_{(x_1, y_1)}}}\right|$$

(iv) The shortest distance between two curves lies along the common normal i.e. the point where tangents are parallel to each other.



Best Practice SHOTS

21. The equation of the normal to the parabola $x^2 = 8y$ at $x = 4$, is [AIEEE 2012]

(a) $x + y = 6$ *(b)* $x + 2y = 0$ *(c)* $3 - 2y = 0$ *(d)* $x + y = 2$

22. The equation of the tangent to the curve $y = x + \frac{4}{x^2}$, that is parallel to the X-axis, is [AIEE 2010]

(a) $y = 2$ *(b)* $y = 3$ *(c)* $y = 0$ *(d)* $y = 1$

23. The curve $\frac{x^n}{a^n} + \frac{y^n}{b^n} = 2$ touches the line $\frac{x}{a} + \frac{y}{b} = 2$ at the point

(a) (b, a) *(b)* (a, b) *(c)* $(1, 1)$ *(d)* $\left(\frac{1}{b}, \frac{1}{a}\right)$

24. Let P be any point on the curve $x^{2/3} + y^{2/3} = a^{2/3}$. Then, the length of the segment of the tangent between the coordinate axes is of length

(a) $3a$ *(b)* $4a$ *(c)* $5a$ *(d)* a

25. If the tangent at any point on the curve $x^4 + y^4 = c^4$ cuts off intercepts a and b on the coordinate axes, the value of $a^{-4/3} + b^{-4/3}$ is

(a) $c^{-4/3}$ *(b)* $c^{-1/2}$ *(c)* $c^{1/2}$ *(d)* None of these

26. For the curve $y = Ce^{x/a}$, which one of the following is incorrect.

(a) subtangent is constant
(b) subnormal varies as the square of the ordinate
(c) tangent at (x_1, y_1) on the curve intersect the X-axis at a distance of $(x_1 - a)$ from the origin
(d) equation of normal at the point where the curve cuts Y-axis is $cy + ax = c$

27. If m is the slope of the tangent to the curve $e^y = 1 + x^2$, then

(a) $|m| > 1$ *(b)* $m < 1$ *(c)* $|m| < 1$ *(d)* $|m| \le 1$

28. The angle between one sided tangents to the curve $y = e^{|x|}$ at $x = 0$ is

(a) $\pi/4$ *(b)* $\pi/6$ *(c)* $\pi/2$ *(d)* $\pi/3$

29. Let $y = f(x)$ be the equation of a parabola which is touched by the line $y = x$ at the point where $x = 1$. Then,

(a) $f'(0) = f'(1)$ *(b)* $f'(1) = -1$
(c) $f(0) + f'(0) + f''(0) = 1$ *(d)* $2f(0) = 1 - f'(0)$

30. If the tangent to the curve $xy + ax + by = 0$ at $(1, 1)$ makes an angle $\tan^{-1} 2$ with X-axis, then $\frac{a+b}{ab}$ is equal to

(a) 0 *(b)* 1/2 *(c)* $-1/2$ *(d)* None of these

31. The equation of the normal to the curve $y = e^{-2|x|}$ at the point where the curve cuts the line $x = -\frac{1}{2}$, is

(a) $2e(ex + 2y) = 4 - e^2$
(b) $2e(ex - 2y) = e^2 - 4$
(c) $2e(ey - 2x) = e^2 - 4$
(d) $2e(ey + 2x) = e^2 - 4$

32. The lengths of tangent, subtangent, normal and subnormal for the curve $y = x^2 + x - 1$ at $(1, 1)$ are A, B, C and D respectively, then their increasing order is

(a) B, D, A, C *(b)* B, A, C, D
(c) A, B, C, D *(d)* B, A, D, C

33. If the line $y = 2x$ touches the curve $y = ax^2 + bx + c$ at the point where $x = 1$ and the curve passes through the point $(-1, 0)$, then

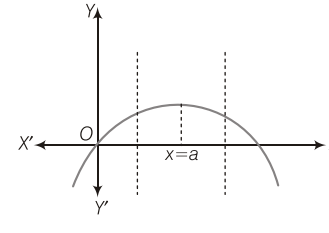
(a) $a = \frac{1}{2}, b = 1, c = \frac{1}{2}$ *(b)* $a = 1, b = \frac{1}{2}, c = \frac{1}{2}$
(c) $a = \frac{1}{2}, c = \frac{1}{2}, b = 1$ *(d)* None of these

MAXIMA AND MINIMA

The maximum/minimum value of a function $f(x)$ implies local or regional maximum/minimum and not the greatest/least value attained by the function. It is also possible in a function that local maximum at one point is smaller than local minimum at another point. Some times we use the word extrema for the maxima and minima.

Maximum

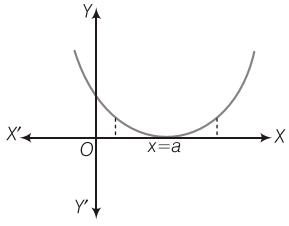
A function $f(x)$ is said to have a maximum or local maximum at $x = a$, if the value of $f(a)$ is greater than all the values of $f(x)$ in a small neighbourhood of $x = a$.



maximum at $x = a$, $f(a - h) < f(a) < f(a + h)$

Minimum

A function $f(x)$ is said to have minimum at $x = a$, if the value of $f(a)$ is less than all values of $f(x)$ is small neighbourhood of $x = a$.



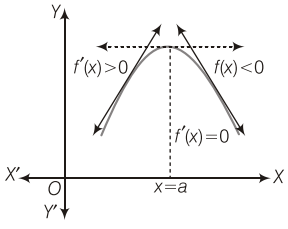
Minimum at $x = 0$, $f(a) < (a - h)$ and $f(a) < (a + h)$

Methods to Find Extremum of a Function

(i) First Derivative Test

Maxima

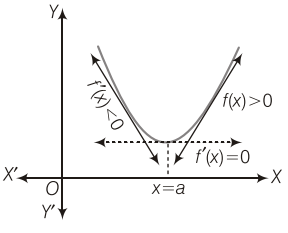
A function $f(x)$ is said to be maximum at $x = a$ if nature of the function changes from increasing to decreasing i.e. $f'(x)$ changes sign from positive to negative at $x = a$ and $f'(a) = 0$.



Thus, $f'(x)$ decreases from positive to zero and then negative, hence $f''(x) < 0$ i.e. $f''(x)$ is negative.

Minima

The function $f(x)$ is said to be minimum at $x = a$ if nature of the function changes from decreasing to increasing as we move from left to right in the neighbourhood of x. i.e. $f'(x)$ changes sign from negative to positive.



Thus, $f'(x)$ increases from negative to zero and then positive, hence $f''(x) > 0$ i.e. $f''(x)$ is positive.

(ii) Second Derivative Test

To find the extremum of a function $f(x)$, we use following steps

***Step* I** Find $f'(x)$ and equate it to zero i.e. $f'(x) = 0$

***Step* II** Solve $f'(x) = 0$, let its roots (critical points) be $x = a_1, a_2, \ldots$

***Step* III** Find $f''(x)$ at $x = a_1$

(a) If $f''(a_1)$ is positive, then $f(x)$ is minimum at $x = a_1$.

(b) If $f''(a_1)$ is negative, then $f(x)$ is maximum at $x = a_1$.

***Step* IV** If at $x = a_1$, $f''(a_1) = 0$, then find $f'''(x)$. If $f'''(a_1) \neq 0$. Then, $f(x)$ is neither maximum nor minimum at $x = a_1$.

***Step* V** If $f'''(a_1) = 0$, then find $f^{iv}(x)$

(a) If $f^{iv}(a_1)$ is positive, then $f(x)$ is minimum at $x = a_1$

(b) If $f^{iv}(a_1)$ is negative, then $f(x)$ is maximum at $x = a_1$

***Step* VI** If at $x = a_1$, $f^{iv}(a_1) = 0$, then find $f^{v}(x)$ proceed same as steps IV and step V. Similarly, check for the other values of x i.e. $a_2, a_3, \ldots$

Concept of Absolute Global Maxima/ Minima

Let $y = f(x)$ be a given function with domain D.

Let $[a, b] \subseteq D$, then maximum/minimum of $f(x)$ in $[a, b]$ is basically the greatest and least value of $f(x)$ in $[a, b]$. Absolute maxima/minima in $[a, b]$ would always occurs at critical points of $f(x)$ with in $[a, b]$ or at the end point of the interval.

Absolute Maximum or Minimum Value of a Function

In order to find absolute maximum or minimum value of a function, we use following steps.

***Step* I** Check the continuity of the function over $[a, b]$

***Step* II** List end points or critical values i.e. $a, c_1, c_2, \ldots, b$

***Step* III** Evaluate $f(a), f(c_1), f(c_2), \ldots, f(b)$

***Step* IV** The absolute maximum value is the greatest value found in step III and the absolute minimum value is smallest values found in step III.

SPEED BOOSTER

(i) The function $y = \ln f(x)$ is maximum or minimum according as $f(x)$ is maximum or minimum.

(ii) The function $y = kf(x)$ or $y = \{f(x)\}^k$ $(k > 0)$ is maximum or minimum according as $f(x)$ is maximum or minimum.

(iii) If $a_1, a_2, a_3, \ldots, a_n = k$ (constant), then the value of $a_1 + a_2 + a_3 + \ldots + a_n$ is least, when $a_1 = a_2 = a_3 = \ldots = a_n$ Hence, least value of $(a_1 + a_2 + a_3 + \ldots + a_n) = n(k)^{1/n}$.

(iv) If $a_1^m + a_2^m + a_3^m + \ldots + a_n^m = k$ and $m \notin (0, 1)$ and the greatest value of $a_1 + a_2 + a_3 + \ldots + a_n$ occurs when $a_1 = a_2 = a_3 = \ldots = a_n$, then the value is $n^{1-\frac{1}{m}} \cdot k^{1/m}$.

MATHS MAXIMA

(i) $y = f(x)$ is maximum or minimum as $\frac{1}{f(x)}$ is maximum or minimum.

(ii) The function $f(x) = \frac{ax + b}{cx + d}$ had no local maximum or minimum regardless of the values of a, b, c and d.

(iii) If the perimeter of rectangles is given, then square will have largest area.

(iv) If area of rectangle is given, then square has the least perimeter.

(v) $f(x) = \sin^m x \cos^n x$ attain maximum value at $x = \tan^{-1}\sqrt{m/n}$..

(vi) A cone of maximum volume that can be inscribed in a sphere of given radius (r) of height $4r/3$ and right circular cylinder of height $2r/\sqrt{3}$.

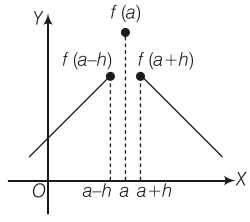
Maxima and Minima of Discontinuous Functions

For discontinuous function, we don't apply the methods discussed earlier. We observe certain conditions and their graphs which would give you more clear picture.

(i) Maximum of Discontinuous Functions

For the maxima of discontinuous functions we have following four cases

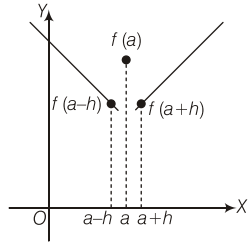
***Case* I**



From this figure,

$f(a) > f(a + h), \quad f(a) > f(a - h)$

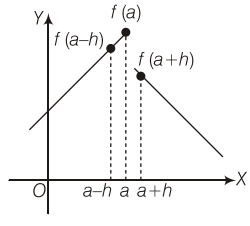
***Case* II**



From this figure,

$f(a) > f(a + h), \quad f(a) > f(a - h)$

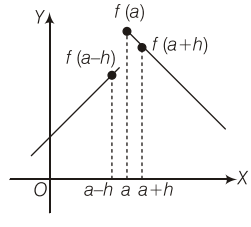
***Case* III**



From this figure,

$f(a) > f(a + h), \quad f(a) \geq f(a - h)$

***Case* IV**



From this figure,

$f(a) \geq f(a + h), \quad f(a) > f(a - h)$

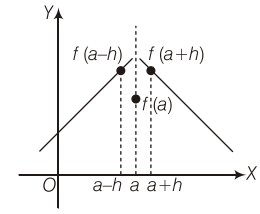
From all the above four mentioned cases for maximum of discontinuous functions, we have

$f(a) \geq f(a + h), f(a) \geq f(a - h)$

(ii) Minimum of Discontinuous Functions

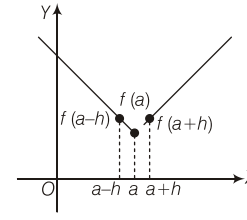
For the minimum of discontinuous functions. The following four cases arises at $x = a$.

***Case* I**



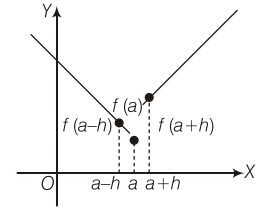
From this figure, $f(a) < f(a + h), \quad f(a) < f(a - h)$

***Case* II**



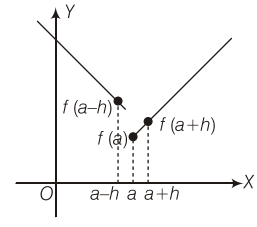
From this figure, $f(a) < f(a + h), \quad f(a) \leq f(a - h)$

***Case* III**



From this figure, $f(a) < f(a + h), \quad f(a) < f(a - h)$

***Case* IV**



From this figure, $f(a) \leq f(a + h), \quad f(a) < f(a - h)$

From all the four above mentioned cases for minimum of discontinuous functions, we have

$f(a) \leq f(a + h)$ and $f(a) \leq f(a - h)$

Properties of Maxima and Minima

(i) If $f(x)$ is continuous function in its domain, then atleast one maxima and one minima must lie between two equal values of x.

(ii) Maxima and minima occur alternatively i.e. between two maxima, there is one minima and *vice-versa*.

(iii) If $f(x) \to \infty$ as $x \to a$ or b and $f'(x) = 0$ only for one value of x (say $c \in (a, b)$), then $f(c)$ is necessarily the minimum and the least value.

(iv) If $f(x) \to -\infty$ as $x \to a$ or b and $f(c)$ is necessarily the maximum and the greatest value.

Best Practice SHOTS

34. Suppose the cubic $x^3 - px + g$ has three real roots where $p > 0$ and $g > 0$. Then, which one of the following holds? **[AIEEE 2008]**

(a) The cubic has minimum at both $\frac{\sqrt{p}}{3}$ and $-\frac{\sqrt{p}}{3}$

(b) The cubic has maximum at both $\frac{\sqrt{p}}{3}$ and $-\frac{\sqrt{p}}{3}$

(c) The cubic has minimum at $\frac{\sqrt{p}}{3}$ and maximum at $-\frac{\sqrt{p}}{3}$

(d) The cubic has minimum at $-\frac{\sqrt{p}}{3}$ and maximum at $\frac{\sqrt{p}}{3}$

35. The critical point of the function $f(x) = (x-2)^{2/3}(2x+1)$ are

(a) 1 and 2 *(b)* 1 and $-1/2$

(c) -1 and 2 *(d)* 1

36. If $f(x)$ is a cubic polynomial which has local maximum at $x = -1$. If $f(2) = 18$, $f(1) = -1$ and $f'(x)$ has local minimum at $x = 0$, then

(a) the distance between $(-1, 2)$ and $(\alpha, f(\alpha))$, where $x = \alpha$ is the point of local minima is $2\sqrt{5}$

(b) $f(x)$ is increasing for $x \in [1, 2\sqrt{5})$ and has a local minima at $x = 1$

(c) the value of $f(0)$ is 5

(d) None of the above

37. The number of values of x where the function $f(x) = \cos x + \cos(\sqrt{2}x)$ attains maximum, is

(a) 0 *(b)* 1

(c) 2 *(d)* infinite

38. The function $f(x) = \frac{x}{1 + x\tan x}$ has

(a) one point of minima in the interval $\left(0, \frac{\pi}{2}\right)$

(b) one point of maxima in the interval $\left(0, \frac{\pi}{2}\right)$

(c) no point of maxima and no point of minima in the interval $\left(0, \frac{\pi}{2}\right)$

(d) two points of maxima in the interval $\left(0, \frac{\pi}{2}\right)$

39. The minimum value of the function $f(x)$ given by $f(x) = \frac{x^m}{m} + \frac{x^{-n}}{n}$, where $\frac{1}{m} + \frac{1}{n} = 1$ and $m > 1$ is

(a) 1
(b) 0
(c) 2
(d) None of the above

40. For $x \in \left(0, \frac{5\pi}{2}\right)$, define $f(x) = \int_0^x \sqrt{t}\sin t\, dt$. Then, f has local maximum at π and 2π

(a) local maximum at π and 2π
(b) local minimum at π and 2π
(c) local minimum at π and maximum at 2π
(d) local maximum at π and minimum at 2π

41. Let $P(x)$ be a real polynomial of least degree which has a local maximum at $x = 1$ and a local minimum at $x = 3$. If $P(1) = 6$ and $P(3) = 2$, then $P'(0)$ is **[IIT 2012]**

(a) 8 *(b)* 9 *(c)* 3 *(d)* 6

42. If $f(x) = \int_0^x e^{t^2}(t-2)(t-3)\, dt$ for all $x \in (0, \infty)$, then which of the following is incorrect? **[IIT 2012]**

(a) f has a local maximum at $x = 2$ and local minimum at $x = 3$
(b) f is decreasing on $(2, 3)$
(c) there exists $c \in (0, \infty)$ such that $f''(c) = 0$
(d) f is increasing on R^+

43. The minimum value of the function $f(x) = 2|x-2| + 5|x-3|$ for all $x \in R$, is

(a) 3 *(b)* 2
(c) 5 *(d)* 7

44. $f(x) = \int_{\cos x}^{\sin x} (1 - t + 2t^3)\, dt$ has in $[0, 2\pi]$

(a) a maximum at $\frac{\pi}{4}$ and a minimum at $\frac{3\pi}{4}$

(b) a maximum at $\frac{3\pi}{4}$ and a minimum at $\frac{7\pi}{4}$

(c) a maximum at $\frac{5\pi}{4}$ and a minimum at $\frac{7\pi}{4}$

(d) neither a maximum nor a minimum

45. The difference between the greatest and least values of the function $f(x) = \int_0^x (6t^2 - 24)\, dt$ or $[1, 3]$ is

(a) 14 *(b)* 10
(c) 4 *(d)* None of these

46. If $P = x^3 - \frac{1}{x^3}$ and $Q = x - \frac{1}{x}$, $x \in (0, \infty)$, then minimum value of $\frac{P}{Q^2}$ is

(a) $2\sqrt{3}$ *(b)* $-2\sqrt{3}$
(c) non-existent *(d)* None of these

47. A function f such that $f'(a) = f''(a) = f'''(a) = \ldots = f^{2n}(a) = 0$ and f has a local maximum value b at $x = a$, if $f(x)$ is

(a) $(x-a)^{2n-2} + b$ *(b)* $b - 1 - (x+a)^{2n+1}$
(c) $b - (x-a)^{2n+2}$ *(d)* $(x-a)^{2n+2} + b$

48. If α and β are respectively the maximum and minimum values of the function $f(x)$ and

$$f(x) = \begin{vmatrix} 1+\sin^2 x & \cos^2 x & \sin 2x \\ \sin^2 x & 1+\cos^2 x & \sin 2x \\ \sin^2 x & \cos^2 x & 1+\sin 2x \end{vmatrix}$$, then which one of the following option is not true?

(a) $\alpha + \beta^{99} = 4$

(b) $\alpha^3 - \beta^{17} = 26$

(c) $\alpha^{2n} - \beta^{2n}$ is always an even integer for $n \in N$

(d) a triangle can be constructed having its sides as α, β and $\alpha - \beta$

49. If the parabola $y = ax^2 + bx + c$ has vertex at (4, 2) and $a \in [1, 3]$, then the difference between the extreme value of abc is equal to

(a) 3600 (b) 144

(c) 3456 (d) None of these

50. A straight line through the point (h, k) where $h > 0$ and $k > 0$, makes positive intercepts on the coordinate axes. Then, the minimum length of the line intercepts between the coordinate axes is

(a) $(h^{2/3} + k^{2/3})^{3/2}$ (b) $(h^{3/2} + k^{3/2})^{2/3}$

(c) $(h^{2/3} - k^{2/3})^{3/2}$ (d) $(h^{3/2} - k^{3/2})^{2/3}$

Answers

1. (a)	**2.** (a)	**3.** (c)	**4.** (c)	**5.** (a)	**6.** (c)	**7.** (a)	**8.** (d)	**9.** (a)	**10.** (b)
11. (c)	**12.** (d)	**13.** (a)	**14.** (a)	**15.** (a)	**16.** (c)	**17.** (b)	**18.** (b)	**19.** (a)	**20.** (b)
21. (a)	**22.** (b)	**23.** (b)	**24.** (d)	**25.** (a)	**26.** (d)	**27.** (d)	**28.** (c)	**29.** (d)	**30.** (b)
31. (a)	**32.** (d)	**33.** (a)	**34.** (c)	**35.** (a)	**36.** (b)	**37.** (b)	**38.** (b)	**39.** (a)	**40.** (d)
41. (b)	**42.** (d)	**43.** (b)	**44.** (b)	**45.** (a)	**46.** (a)	**47.** (c)	**48.** (d)	**49.** (c)	**50.** (a)

Formula at a Glance

1. The equation of the tangent to a curve at (x_0, y_0)

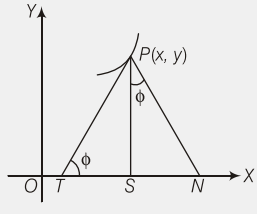
$$y - y_0 = m(x - x_0)$$

The equation of the normal to a curve at (x_0, y_0)

$$y - y_0 = -\frac{1}{m}(x - x_0)$$

where $$m = \left[\frac{dy}{dx}\right]_{\text{at } (x_0, y_0)}$$

2. Length of the tangent $= PT = y \operatorname{cosec} \phi = y\sqrt{1 + \left(\frac{dx}{dy}\right)^2}$



Length of the subtangent $= TS = y \cot \phi = y\frac{dx}{dy}$

Length of the normal $= PN = y \sec \phi = y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$

Length of the subnormal $= SN = y \tan \phi = y\frac{dy}{dx}$

3. The angle of intersection of two curves $\tan \phi = \left|\frac{m_2 - m_1}{1 + m_1 m_2}\right|$

where m_1, m_2 are the slops of the tangents to the curves at the point of intersection.

4. Increasing and decreasing functions

- If $f(x_2) > f(x_1)$, $\forall\ x_2 > x_1$; x_1, $x_2 \in D_f$, then $f(x)$ is strictly increasing.
- If $f'(x) > 0, \forall\ x \in D_f$, then $f(x)$ is strictly increasing function.
- If $f(x_2) \geq f(x_1)$, $\forall x_2 > x_1$; x_1, $x_2 \in D_f$, then $f(x)$ is increasing function.
- If $f'(x) \geq 0, \forall\ x \in D_f$, then $f(x)$ is increasing function.
- If $f(x_2) < f(x_1), \forall\ x_2 > x_1$; x_1, $x_2 \in D_f$, then $f(x)$ is strictly decreasing function.
- If $f'(x) < 0, \forall\ x \in D_f$, then $f(x)$ is strictly decreasing function.
- If $f(x_2) \leq f(x_1), \forall\ x_2 > x_1$; x_1, $x_2 \in D_f$, then $f(x)$ is decreasing function.
- If $f'(x) \leq 0, \forall\ x \in D_f$, then $f(x)$ is decreasing function.

5. **Maxima and Minima**

- If $f(x_0 + h) > f(x_0)$ and $f(x_0 - h) > f(x_0)$, then $f(x)$ has a relative minima at $x = x_0$.
- If $f(x_0 + h) < f(x_0)$ and $f(x_0 - h) < f(x_0)$, then $f(x)$ has a relative maxima at $x = x_0$.
- If $f(x)$ is differentiable function, then for maxima or minima $f'(x) = 0 \Rightarrow x = x_1, x_2, \ldots, x_k, \ldots$
- **First derivative test** If f' changes its sign from positive to negative about a critical point then it has point of local maxima.
- If f' changes sign from negative to positive, then f has a point of minima.
- If there is no sign change about the critical point, then f has neither maxima nor minima at that point.
- ***n*th derivative test** Solve $f'(x) = 0 \Rightarrow x = x_1, x_2, \ldots, x_k, \ldots$
- If $f''(x_k) > 0$, then $f(x)$ has relative minima at $x = x_k$.
- If $f''(x_k) < 0$, then $f(x)$ has relative maxima at $x = x_k$.
- If $f''(x_k) = 0$, find $f'''(x)$.
- If $f'''(x_k) \neq 0 \Rightarrow$ neither maxima nor minima it is called point of inflection.
- If $f'''(x_k) = 0$, find $f''''(x)$.

Repeat the process till we get $f^{(n)}(x_k) \neq 0$.

- If n is odd $\Rightarrow$ neither maxima nor minima
- If n is even $\Rightarrow f^{(n)}(x_k) > 0 \Rightarrow$ minima at $x = x_k$.
  $f^{(n)}(x_k) < 0 \Rightarrow$ maxima at $x = x_k$.

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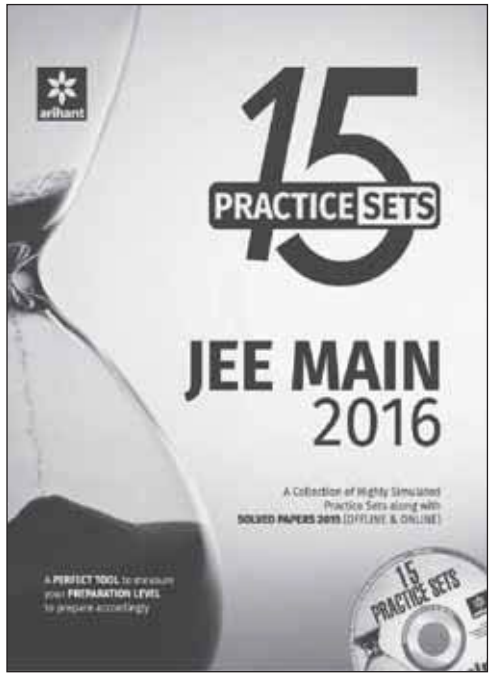
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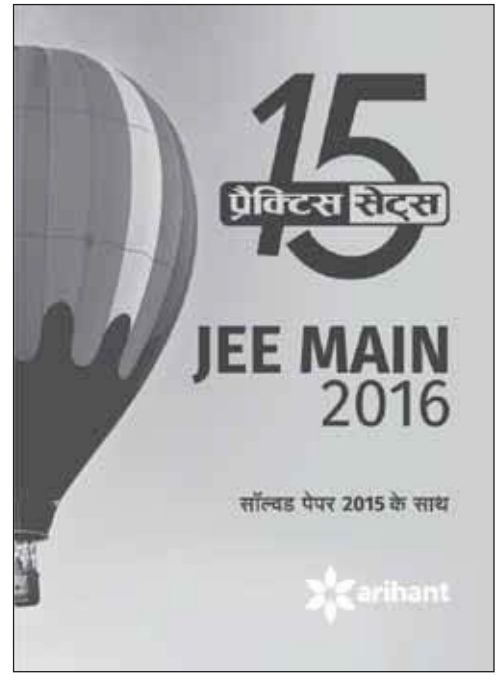

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MATHEMATICS SPECTRUM *for You*

In this section, we invite our readers to send their problems and we will try to provide the best possible solutions given by our expert panel.

Q1 Sir, I am unable to crack the following sum $\lim_{x\to 0}\dfrac{\log\log(1-x^2)}{\log\log\cos x}$

Please provide me its solution with detailed explanation.

[Ansh Chopra, Lakhimpur]

➥ Dear Ansh, when we substitute $x = 0$ in the given problem we get an indeterminate form of type $\left(\dfrac{\infty}{\infty}\right)$

In this case, we use L' hospital's rule, so you should aware of L' hospital's rule. This rule states that

If $f(x)$ and $g(x)$ are two functions of x such that

(i) $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$ or ∞

(ii) both are continuous and differentiable at $x = a$

(iii) $f'(x)$ and $g'(x)$ are continuous at $x = a$, then

$\lim_{x\to a}\dfrac{f(x)}{g(x)} = \lim_{x\to a}\dfrac{f'(x)}{g'(x)}$ provided $g'(a) \neq 0$

Now, $\lim_{x\to 0}\dfrac{\log\log(1-x^2)}{\log\log\cos x}$ $\left[\text{form}\left(\dfrac{\infty}{\infty}\right)\right]$

Differentiate N^r or D^r separately w.r.t. x,

$$= \lim_{x\to 0}\frac{\dfrac{1}{\log(1-x^2)}\times\dfrac{1}{(1-x^2)}\times(-2x)}{\dfrac{1}{\log\cos x}\cdot\dfrac{1}{\cos x}(-\sin x)}$$

$$= 2\lim_{x\to 0}\frac{x\cos x\log\cos x}{(1-x^2)\log(1-x^2)\sin x}$$

$$= 2\lim_{x\to 0}\frac{x}{\sin x}\cdot\lim_{x\to 0}\frac{\cos x}{1-x^2}\cdot\lim_{x\to 0}\frac{\log\cos x}{\log(1-x^2)}$$

$$= 2\times 1\times 1\lim_{x\to 0}\frac{\log\cos x}{\log(1-x^2)} \qquad \left[\text{form}\left(\frac{0}{0}\right)\right]$$

Again differentiating N^r and D^r w.r.t. x,

$$\lim_{x\to 0}\frac{\log\log(1-x^2)}{\log\log\cos x} = 2\lim_{x\to 0}\frac{\dfrac{1}{\cos x}\cdot(-\sin x)}{\dfrac{1}{(1-x^2)}(-2x)}$$

$$= 2\times\frac{1}{2}\lim_{x\to 0}\left(\frac{\sin x}{x}\cdot\frac{1-x^2}{\cos x}\right) = 1$$

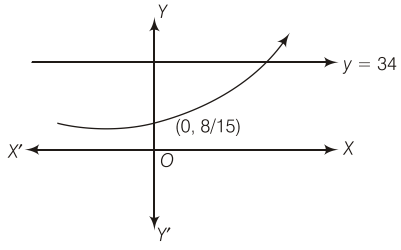
Hence, $\lim_{x\to 0}\dfrac{\log\log(1-x^2)}{\log\log\cos x} = 1$

Q2 Sir, please provide me a shortcut to find the solution of following sum.

"How many solutions are there of the equation $3^{x-1}+5^{x-1}=34$"? [Dollar Imphal]

➥ Dear Dollar, the graphical approach to find the number of solution of any equation would be shortest.

So, first of all we will draw the graph of $y = 3^{x-1} + 5^{x-1}$ and $y = 34$. The points of intersection of these two graphs will be the number of solution of the given equation.



From the graph, it is clear that these two curves intersect at exactly one point.

Hence, it has exact one solution.

Q3 Sir, I am facing problem in solving the following question.

"Message, are conveyed by arranging four white, one blue and three red flags on a pole. Flags of the same colour are alike. If a message is transmitted by the order in which the colours are arranged. Find the total number of messages that can be transmitted, if exactly six flags are used." **[Shashank Ahluvalia, Ghaziabad]**

➥ Dear Shashank, according to the information displayed in the question, we have following cases

Cases	Flags	Number of signals
4 alike and 2 other alike	4 white and 2 red	$\frac{6!}{4!2!} = 15$
4 alike and 2 other different	4 white, 1 red and 1 blue	$\frac{6!}{4!} = 30$
3 alike and 3 other alike	3 white, 3 red	$\frac{6!}{3!3!} = 20$
3 alike, 2 other alike and 1 different	3 white, 1 blue, 2 red or 3 red ,1 blue, 2 white	${}^2C_1 \times \frac{6!}{3!2!} = 120$
		Total = 185

Q4 Sir, please provide me the best solution of the following question.

"If A and B are two independent events, then the probability that both A and B occur is 1/8 and the probability that neither of them occur is 3/8. Find the probability of occurrence of A." **[Bandana Sengar, Jhansi]**

➥ Dear Bandana, first of all you should aware of the properties of independent events.

If A and B are independent events, then

$$P(A \cap B) = P(A) \cdot P(B)$$

likewise $P(\bar{A} \cap \bar{B}) = P(\bar{A}) \cdot P(\bar{B})$

Here, $P(A \cap B) = \frac{1}{8} = P(A) \cdot P(B)$ ...(i)

and $P(\bar{A} \cap \bar{B}) = P(\bar{A}) = P(\bar{B}) = \frac{3}{8}$ ...(ii)

Also, we know $P(\bar{A} \cap \bar{B}) = P(\overline{A \cup B}) \Rightarrow \frac{3}{8} = 1 - P(A \cup B)$

$\Rightarrow P(A \cup B) = \frac{5}{8}$

$\Rightarrow P(A) + P(B) - P(A \cap B) = \frac{5}{8}$

$\Rightarrow P(A) + P(B) - \frac{1}{8} = \frac{5}{8}$

$\Rightarrow P(A) + P(B) = \frac{6}{8}$ ...(iii)

The quadratic equation whose roots are $P(A)$ and $P(B)$, is

$$x^2 - [P(A) + P(B)]x + P(A) \cdot P(B) = 0$$

$\Rightarrow x^2 - \frac{6}{8}x + \frac{1}{8} = 0$ [using Eqs. (i) and (iii)]

$\Rightarrow 8x^2 - 6x + 1 = 0 \Rightarrow x = \frac{1}{2}, \frac{1}{4}$

Then, either $P(A) = \frac{1}{2}, P(B) = \frac{1}{4}$ or $P(A) = \frac{1}{4}, P(B) = \frac{1}{2}$

Q5 Sir, I am unable to solve the following problem.

"If $a = \cos\theta + i\sin\theta$, $b = \cos 2\theta - i\sin 2\theta$, $c = \cos 3\theta + i\sin 3\theta$ and $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 0$, then find the value of θ. Please provide me the best solution".

[Sunil Sharma, Kanpur]

➥ Dear Sunil, first of all simplify the given determinant, after that substitute the values of a, b and c to solve for θ.

Given, $\Delta = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 0$

$\Rightarrow (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca) = 0$

$\Rightarrow \frac{1}{2}(a + b + c)[(a - b)^2 + (b - c)^2 + (c - a)^2] = 0$

$\Rightarrow a + b + c = 0$ or $(a - b)^2 + (b - c)^2 + (c - a)^2 = 0$

$\Rightarrow a + b + c = 0$ or $a = b = c$

If $a + b + c = 0$, then we have

$\cos\theta + \cos 2\theta + \cos 3\theta = 0$ and

$\sin\theta - \sin 2\theta + \sin 3\theta = 0$

$\cos 2\theta (2\cos\theta + 1) = 0$ and $\sin 2\theta (1 - 2\cos\theta) = 0$ ...(i)

which is not possible as $\cos 2\theta = 0$, gives $\sin 2\theta \neq 0$, $\cos\theta \neq \frac{1}{2}$

and $\cos\theta = -\frac{1}{2}$ gives $\sin 2\theta \neq 0$, $\cos\theta \neq \frac{1}{2}$

Therefore Eq. (i) does not hold simultaneously

$\therefore a + b + c \neq 0$

$\Rightarrow a = b = c$ or $e^{i\theta} = e^{-2i\theta} = e^{3i\theta}$

which is satisfied only by $e^{i\theta} = 1$

i.e. $\cos\theta = 1, \sin\theta = 0$

So, $\theta = 2k\pi, k \in Z$

SHORTCUT Methods

Here, we are introducing some important shortcut methods on integration to save your precious time in tedious calculations based problem asked in various engineering entrance exams. Use these shortcuts, to practice questions and be acquainted with it.

1. $\int \sin^n x\,dx = \dfrac{-\sin^{n-1}x\cos x}{n} + \dfrac{n-1}{n}\int \sin^{n-2}x\,dx$

Example $\int \sin^5 x\,dx = \dfrac{-\sin^4 x\cdot\cos x}{5} + \dfrac{4}{5}\int \sin^3 x\,dx$

$= \dfrac{-\sin^4 x\cdot\cos x}{5} + \dfrac{4}{5}\left[\dfrac{-\sin^2 x\cos x}{3} - \dfrac{2}{3}\cos x + C\right]$

$= \dfrac{-\sin^4 x\cdot\cos x}{5} - \dfrac{4\sin^2 x\cos x}{15} - \dfrac{8}{15}\cos x + C_1$

2. $\int \cos^n x\,dx = \dfrac{\cos^{n-1}x\cdot\sin x}{n} + \dfrac{n-1}{n}\int \cos^{n-2}x\,dx$

Example $\int \cos^3 x\,dx = \dfrac{\cos^2 x\sin x}{3} + \dfrac{2}{3}\int \cos x\,dx$

$= \dfrac{\cos^2 x\sin x}{3} + \dfrac{2}{3}\sin x + C = \dfrac{\sin 2x\cdot\cos x}{6} + \dfrac{2}{3}\sin x + C$

3. $\int \tan^n x\,dx = \dfrac{\tan^{n-1}x}{n-1} - \int \tan^{n-2}x\,dx$

Example $\int \tan^4 x\,dx = \dfrac{\tan^3 x}{3} - \int(\sec^2 x - 1)\,dx$

$= \dfrac{\tan^3 x}{3} - \tan x + x + C$

4. $\int \dfrac{a\cos x + b\sin x}{c\cos x + d\sin x} = \dfrac{ac+bd}{c^2+d^2}x + \dfrac{ad-bc}{c^2+d^2}\log|c\cos x + d\sin x| + C$

Example $\int \dfrac{2\cos x + 3\sin x}{4\cos x + 5\sin x}\,dx$

$= \dfrac{23}{41}x - \dfrac{2}{41}\log|4\cos x + 5\sin x| + C$

5. (a) If $a > b$, then

$$\int \frac{dx}{a+b\cos x} = \frac{1}{\sqrt{a^2-b^2}}\tan^{-1}\left(\sqrt{\frac{a-b}{a+b}}\tan\frac{x}{2}\right) + C$$

(b) If $a < b$, then

$$\int \frac{dx}{a+b\cos x} = \frac{1}{\sqrt{b^2-a^2}}\log\left|\frac{\sqrt{b+a}+\sqrt{b-a}\tan\frac{x}{2}}{\sqrt{b+a}-\sqrt{b-a}\tan\frac{x}{2}}\right| + C$$

Example $\int \dfrac{dx}{4+3\cos x}$

$= \dfrac{1}{\sqrt{16-9}}\tan^{-1}\left(\sqrt{\dfrac{4-3}{4+3}}\tan\dfrac{x}{2}\right) + C$

$= \dfrac{1}{\sqrt{7}}\tan^{-1}\left(\dfrac{1}{\sqrt{7}}\tan\dfrac{x}{2}\right) + C$

Example $\int \dfrac{dx}{3+4\cos x}$

$= \dfrac{1}{\sqrt{16-9}}\log\left(\dfrac{\sqrt{4+3}+\sqrt{4-3}\tan\frac{x}{2}}{\sqrt{4+3}-\sqrt{4-3}\tan\frac{x}{2}}\right) + C$

$= \dfrac{1}{\sqrt{7}}\log\left(\dfrac{\sqrt{7}+\tan\frac{x}{2}}{\sqrt{7}-\tan\frac{x}{2}}\right) + C$

6. If $0 < a < b$, then $\int \dfrac{dx}{a+b\sin x}$

$$= \frac{1}{\sqrt{b^2-a^2}}\log\left|\frac{a\tan\frac{x}{2}+b-\sqrt{b^2-a^2}}{a\tan\frac{x}{2}+b+\sqrt{b^2-a^2}}\right| + C$$

Example $\int \dfrac{dx}{3+5\sin x}$

$= \dfrac{1}{\sqrt{25-9}}\log\left|\dfrac{3\tan\frac{x}{2}+5-\sqrt{25-9}}{3\tan\frac{x}{2}+5+\sqrt{25-9}}\right| + C = \dfrac{1}{4}\log\left|\dfrac{3\tan\frac{x}{2}+1}{3\tan\frac{x}{2}+9}\right| + C$

7. If $b^2 - 4ac < 0$, then

$$\int \frac{dx}{ax^2 + bx + C} = \frac{2}{4ac - b^2} \tan^{-1}\left(\frac{2ax + b}{\sqrt{4ac - b^2}}\right) + C$$

Example $\int \frac{dx}{2x^2 + x + 1}$

$$= \frac{2}{\sqrt{8-1}} \tan^{-1}\left(\frac{4x+1}{\sqrt{8-1}}\right) + C = \frac{2}{\sqrt{7}} \tan^{-1}\left(\frac{4x+1}{\sqrt{7}}\right) + C$$

8. If $b^2 - 4ac < 0$, then

$$\int \frac{(Ax + B)dx}{ax^2 + bx + c} = \frac{A}{2a} \log|ax^2 + bx + c| + \frac{2(2aB - bA)}{A\sqrt{4ac - b^2}} \tan^{-1}\left(\frac{2ax + b}{\sqrt{4ac - b^2}}\right) + C$$

Example $\int \frac{(3x+5)\,dx}{x^2 + 4x + 6} = \frac{3}{2\times 1} \log|x^2 + 4x + 6|$

$$+ \frac{2\,(2\times 1\times 5 - 4\times 3)}{3\sqrt{4\times 1\times 6 - 16}} \cdot \tan^{-1}\left(\frac{2\times 1\times x + 4}{\sqrt{4\times 1\times 6 - 16}}\right) + C$$

$$= \frac{3}{2} \log|x^2 + 4x + 6| - \frac{4}{3\sqrt{8}} \tan^{-1}\frac{2x+4}{\sqrt{8}} + C$$

9. If $a < 0$, $b^2 - 4ac > 0$, then

$$\int \frac{dx}{\sqrt{ax^2 + bx + c}} = \frac{1}{\sqrt{-a}} \sin^{-1}\left(\frac{-2ax - b}{\sqrt{b^2 - 4ac}}\right) + C$$

Example $\int \frac{dx}{\sqrt{5 + 4x - 3x^2}}$

$$= \frac{1}{\sqrt{-(-3)}} \sin^{-1}\left[\frac{-2(-3)x - 4}{\sqrt{76}}\right] + C$$

$$= \frac{1}{\sqrt{3}} \sin^{-1}\left(\frac{6x-4}{\sqrt{76}}\right) + C$$

10. $\int \frac{Ax + B}{\sqrt{ax^2 + bx + c}}\,dx = \frac{A}{a}\sqrt{ax^2 + bx + c} + \frac{2aB - bA}{2a} \int \frac{dx}{\sqrt{ax^2 + bx + c}}$

Example $\int \frac{3x+5}{\sqrt{x^2 + 2x + 4}}\,dx$

$$= \frac{3}{1}\sqrt{x^2 + 2x + 4} + \frac{2\times 5 - 2\times 3}{2}$$

$$\int \frac{dx}{\sqrt{x^2 + 2x + 4}} = 3\sqrt{x^2 + 2x + 4} + 2\int \frac{dx}{\sqrt{x^2 + 2x + 1 + 3}}$$

$$= 3\sqrt{x^2 + 2x + 4} + 2\log\left[(x+1) + \sqrt{x^2 + 2x + 4}\right] + C$$

11. $\int \sqrt{ax^2 + bx + c}\,dx = \frac{(2ax + b)\sqrt{ax^2 + bx + c}}{4a} + \frac{4ac - b^2}{8a} \int \frac{dx}{\sqrt{ax^2 + bx + c}}$

Example $\int \sqrt{-3x^2 + x + 4}\,dx$

$$= \frac{(-6x+1)\sqrt{-3x^2 + x + 4}}{-12} + \left(\frac{-48-1}{-24}\right) \int \frac{dx}{\sqrt{-3x^2 + x + 4}}$$

$$= \frac{(6x-1)\sqrt{-3x^2 + x + 4}}{12} + \frac{49}{24} \int \frac{dx}{\sqrt{-3x^2 + x + 4}}$$

$$= \frac{(6x-1)\sqrt{-3x^2 + x + 4}}{12} + \frac{49}{24}\frac{1}{\sqrt{3}} \sin^{-1}\left(\frac{6x-1}{\sqrt{1+48}}\right) + C$$

$$= \frac{(6x-1)\sqrt{-3x^2 + x + 4}}{12} + \frac{49}{24\sqrt{3}} \sin^{-1}\left(\frac{6x-1}{7}\right) + C$$

12. $\int [xf'(x) + f(x)]\,dx = x\,f(x) + C$

Example $\int [x \cdot \cos x + \sin x]\,dx$

$$= x \sin x + C$$

Here, $f(x) = \sin x$ and $f'(x) = \cos x$

13. $\int e^{ax}\left[f(x) + \frac{f'(x)}{a}\right]dx = \frac{e^{ax} f(x)}{a} + C$

Example $\int e^{5x}\left[x^3 + \frac{3x^2}{5}\right]dx$

$$= \frac{e^{5x} \cdot x^3}{5} + C$$

14. $\int e^x\,[f(x) - f''(x)]\,dx = e^x\,[f(x) - f'(x)] + C$

Example $\int e^x[4x^3 - 24x]\,dx$

$$= e^x\,[4x^3 - 12x^2] + C$$

15. $\int e^x[f(x) + f'''(x)]\,dx = e^x\,\{f(x) - f'(x) + f''(x)\} + C$

Example $\int e^x\,(\sin x - \cos x)\,dx$

$$= e^x\,[\sin x + \sin x - \cos x] + C = e^x\,[2\sin x - \cos x] + C$$

16. $\int e^{f(x)}\left[xf'(x) + \frac{f''(x)}{(f'(x))^2}\right] = e^{f(x)}\left[x - \frac{1}{f'(x)}\right] + C$

Example $\int e^{\cos x}\left[x \cdot (-\sin x) + \frac{(-\cos x)}{(-\sin x)^2}\right]$

$$= e^{\cos x}\left[x + \frac{1}{\sin x}\right] + C$$

17. $\int e^{f(x)} \cdot [g(x) \cdot f'(x) + g'(x)]\,dx = e^{f(x)}\,g(x) + C$

Example $\int e^{\tan^{-1} x}\left[\frac{\cos x}{1 + x^2} - \sin x\right]dx$

$$= e^{\tan^{-1} x} \cdot \cos x + C$$

18. $$\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{(a^2+b^2)}(a\cos bx + b\sin bx) + C$$
$$= \frac{e^{ax}}{\sqrt{a^2+b^2}}\cos\left(bx - \tan^{-1}\frac{b}{a}\right) + C$$

Example $$\int e^{3x}\cos 4x\,dx$$
$$= \frac{e^{3x}}{5}\cos\left(4x - \tan^{-1}\frac{4}{3}\right) + C$$

19. $$\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx) + C$$
$$= \frac{e^{ax}}{\sqrt{a^2+b^2}}\sin\left[bx - \tan^{-1}\left(\frac{b}{a}\right)\right] + C$$

Example $$\int e^{4x}\sin 3x\,dx$$
$$= \frac{e^{4x}}{5}\sin\left[3x - \tan^{-1}\left(\frac{3}{4}\right)\right] + C$$

20. (a) $$\int x\sin^n x\,dx = \frac{-x\cos x\cdot\sin^{n-1}x}{n} + \frac{\sin^n x}{n^2} + \frac{n-1}{n}\cdot\int x\cdot\sin^{n-2}x\,dx$$

(b) $$\int x\cos^n x\,dx = \frac{-x\sin x\cdot\cos^{n-1}x}{n} + \frac{\cos^n x}{n^2} + \frac{n-1}{n}\cdot\int x\cdot\cos^{n-2}x\,dx$$

Example
$$\int x\sin^2 x\,dx = \frac{-x\cos x\cdot\sin x}{2} + \frac{\sin^2 x}{4} + \frac{1}{2}\int x\,dx$$
$$= \frac{-x\sin 2x}{4} + \frac{\sin^2 x}{4} + \frac{x^2}{4} + C$$

Example $$\int x\cos^2 x\,dx = \frac{-x\cos x\cdot\sin x}{2} + \frac{\cos^2 x}{4} + \frac{1}{2}\int x\,dx$$
$$= \frac{-x\sin 2x}{4} + \frac{\cos^2 x}{4} + \frac{x^2}{4} + C$$

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4. *There is no overall choice. However, internal choice has been provided in 4 questions of four marks each and 2 questions of six marks each. You have to attempt only one of the alternatives in all such questions.*
5. *Use of calculators is not permitted. You may ask for logarithmic tables, if required.*

Section A

1. If $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ are unit vectors, then what is the angle between $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ for $(\sqrt{3}\ \hat{\mathbf{a}} - \hat{\mathbf{b}})$ to be a unit vector?

2. Find a vector in the direction of $\mathbf{a} = 4\hat{\mathbf{i}} - \hat{\mathbf{j}} + 3\hat{\mathbf{k}}$. whose magnitude is 3.

3. Find the intercepts made by the plane $2x - 3y + 5z + 4 = 0$ on the coordinate axis.

4. Find the solution of the differential equation $y^2 \frac{dy}{dx} + y^2 + 1 = 0$.

5. Find the differential equation corresponding to $y^2 = a(b - x)(b + x)$ by eliminating parameters a and b.

6. If A and B are symmetric matrices of the same, order, then show that $(AB' - BA')$ is a skew-symmetric matrix.

Section B

7. If $A = \begin{bmatrix} 1 & 5 \\ 7 & 12 \end{bmatrix}$ and $B = \begin{bmatrix} 9 & 1 \\ 7 & 8 \end{bmatrix}$, then find a matrix C such that $3A + 5B + 2C$ is a null matrix.

or

Find x, y and z, if $A = \begin{bmatrix} 0 & 2y & z \\ x & y & -z \\ x & -y & z \end{bmatrix}$ satisfies $A' = A^{-1}$.

8. Evaluate $\int \frac{4x + 5}{\sqrt{2x^2 + x - 3}}\, dx$.

or

Evaluate $\int \sqrt{\frac{1 - \sqrt{x}}{1 + \sqrt{x}}}\, dx$.

9. Prove that

(i) $P(A) = P(A \cap B) + P(A \cap \overline{B})$

(ii) $P(A \cup B) = P(A \cap B) + P(A \cap \overline{B}) + P(\overline{A} \cap B)$

10. Find a vector of magnitude 5 units, perpendicular to each of the vectors $(\mathbf{a} + \mathbf{b})$ and $(\mathbf{a} - \mathbf{b})$, where $\mathbf{a} = \hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$ and $\mathbf{b} = \hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$.

11. If $x = \sqrt{a^{\sin^{-1} t}}$ and $y = \sqrt{a^{\cos^{-1} t}}$, then show that $\frac{dy}{dx} = -\frac{y}{x}$.

12. By computing the shortest distance determine whether the following pairs of lines intersect or not $\frac{x-1}{2} = \frac{y+1}{3} = z$; $\frac{x+1}{5} = \frac{y-2}{1}$, $z = 2$

13. If $\tan^{-1}\frac{x-1}{x-2}+\tan^{-1}\frac{x+1}{x+2}=\frac{\pi}{4}$, then find the value of x.

or

Prove that $\tan^{-1}\frac{63}{16}=\sin^{-1}\frac{5}{13}+\cos^{-1}\frac{3}{5}$.

14. If $a+b+c\neq 0$ and $\begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}=0$, then prove that $a=b=c$.

15. Evaluate $\int_0^1 \frac{\log(1+x)}{1+x^2}\,dx$.

16. Two men A and B start with velocities v at the same time from the junction of two roads inclined at 45° to each other. If they travel by different roads, then find the rate at which they are being separated.

17. Prove that

$$\frac{d}{dx}\left[\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right]=\sqrt{a^2-x^2}.$$

or

If $y=\{\log(x+\sqrt{x^2+1}\}^2$, then show that $(1+x^2)\,y_2+xy_1=2$.

Section C

18. Evaluate $\int \frac{(\cos 5x+\cos 4x)}{1-2\cos 3x}\,dx$.

19. Two institutions decided to award their employees for the three values of resourcefulness, competence and determination in the form of prizes at the rate of ₹ x, ₹ y and ₹ z respectively per person. The first institution decided to award respectively 4, 3 and 2 employees with a total prize money of ₹ 37000 and the second institution decided to award respectively 5, 3 and 4 employees with a total prize money of ₹ 47000. If all the three prizes per person together amount to ₹ 12000, then using matrix method, find the values of x, y and z. What values are described in this question?

20. Let $A=\{x\in Z: 0\le x\le 12\}$.

Show that $R=\{(a,b): |a-b| \text{ is a multiple of } 4\}$ is (i) reflexive (ii) symmetric and (iii) transitive.

Find the set of elements related to 1.

21. Find the area of the region included between the parabolas $y^2=4ax$ and $x^2=4ay$, where $a>0$.

or

Evaluate $\int_0^1 (3x^2+2x+1)\,dx$ as limit of sums.

22. Find the particular solution of the differential equation $(\tan^{-1}y-x)\,dy=(1+y^2)\,dx$, given that when $x=0, y=0$.

or

Find the equation of the curve that passes through the point (1,2) and satisfies the differential equation $\frac{dy}{dx}=\frac{-2xy}{(x^2+1)}$.

23. Find the length and the foot of the perpendicular from the point $P(7,14,5)$ to the plane $2x+4y-z=2$. Also, find the image of the point P in the plane.

24. A problem in Mathematics is given to three students whose chances of solving it correctly are $\frac{1}{2}$, $\frac{1}{3}$ and $\frac{1}{4}$ respectively. What is the probability that only one of them solves it correctly?

25. Show that of all the rectangles inscribed in a given fixed circle, the square has the maximum area.

26. A firm manufactures two types of products, A and B and sells them at a profit of ₹ 5 per unit of type A and ₹ 3 per unit of type B. Each product is processed on two machines M_1 and M_2. One unit of type A requires one minute of processing time on M_1 and two minutes of processing time on M_2, whereas one unit of type B requires one minute of processing time on M_1 and one minute on M_2. Machines M_1 and M_2 are respectively available for atmost 5 h and 6 h in a day. Find out how many units of each type of product should the firm produce a day in order to maximise the profit. Solve the problem graphically.

Notice Board

AMUEEE will be on April 24, 2016

Aligarh Muslim University has announced dates for Aligarh Muslim University Engineering Entrance Exam (AMUEEE) for the academic year 2016-17. For providing admissions to undergraduate degree courses in engineering (B. Tech and B. Arch programmes) offered by various colleges under the University, AMUEEE 2016 Exam will be conducted on April 24, 2016. Aspiring candidates, who wish to get admissions in undergraduate engineering courses at various colleges under Aligarh Muslim University, are required to appear in AMUEEE 2016. Various details about the exam are given below for students'

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Section A

1. Find the projection of $\mathbf{a}$ on $\mathbf{b}$ if $\mathbf{a}\cdot\mathbf{b}=8$ and $\mathbf{b}=2\hat{\mathbf{i}}+6\hat{\mathbf{j}}+3\hat{\mathbf{k}}$.

2. If $|\mathbf{a}|=3$ and $-1\le k\le 2$, then find the interval in which $|k\mathbf{a}|$ will lie.

3. Find the integrating factor of the differential equation $x\dfrac{dy}{dx}-y=\sin x$.

4. Find the total number of possible matrices of order 3×3 with each entry 2 or 0.

5. Find the differential equation representing the family of curves $y = A\cos(x + B)$, where A and B are parameters.

6. If a line makes angles $\dfrac{\pi}{2}, \dfrac{3}{4}\pi$ and $\dfrac{\pi}{4}$ with X, Y, Z-axes, respectively, then find its direction cosines.

Section B

7. If $A=\begin{bmatrix}3 & -5\\ -4 & 2\end{bmatrix}$, then find $A^2-5A-14I$. Hence, obtain A^3.

or

If possible, using elementary row transformations, find the inverse of the given matrix.

$$\begin{bmatrix}2 & -1 & 3\\ -5 & 3 & 1\\ -3 & 2 & 3\end{bmatrix}$$

8. Evaluate $\int \dfrac{x^3+x+1}{x^2-1}\,dx$.

or

Evaluate $\int \dfrac{2}{(1-x)(1+x^2)}\,dx$.

9. Two dice are thrown together and the total score is noted. The events E, F and G are 'a total of 4' , 'a total of 9 or more' and 'a total divisible by 5', respectively. Calculate $P(E)$, $P(F)$ and $P(G)$ and decide which pairs of events, if any are independent.

10. Using vectors find the area of triangle with vertices $A(1, 1, 2)$, $B(2, 3, 5)$ and $C(1, 5, 5)$.

or

If $\begin{vmatrix}a & a^2 & 1+a^3\\ b & b^2 & 1+b^3\\ c & c^2 & 1+c^3\end{vmatrix}=0$ and the vectors $\mathbf{A}=\hat{\mathbf{i}}+a\hat{\mathbf{j}}+a^2\hat{\mathbf{k}}$, $\mathbf{B}=\hat{\mathbf{i}}+b\hat{\mathbf{j}}+b^2\hat{\mathbf{k}}$, $\mathbf{C}=\hat{\mathbf{i}}+c\hat{\mathbf{j}}+c^2\hat{\mathbf{k}}$ are non-coplanar, then prove that $abc=-1$.

11. Show that $\sin[\cot^{-1}\{\cos(\tan^{-1} x)\}] = \sqrt{\frac{x^2+1}{x^2+2}}$.

or

Prove the following

$$\cos\left(\sin^{-1}\frac{3}{5} + \cot^{-1}\frac{3}{2}\right) = \frac{6}{5\sqrt{13}}$$

12. If $A = \begin{vmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{vmatrix}$ and $B = \begin{vmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{vmatrix}$, then find BA and use this to solve the system of equations $y + 2z = 7$, $x - y = 3$ and $2x + 3y + 4z = 17$.

13. If $x = a(\cos t + t \sin t)$ and $y = a(\sin t - t \cos t)$, $0 < t < \frac{\pi}{2}$, then find $\frac{d^2x}{dt^2}$, $\frac{d^2y}{dt^2}$ and $\frac{d^2y}{dx^2}$.

14. Prove that $\int_0^a \sin^{-1}\left(\sqrt{\frac{x}{a+x}}\right) dx = \frac{a}{2}(\pi - 2)$.

15. Find the shortest distance between the following lines whose vector equations are

$$\mathbf{r} = (1-t)\hat{\mathbf{i}} + (t-2)\hat{\mathbf{j}} + (3-2t)\hat{\mathbf{k}} \text{ and}$$
$$\mathbf{r} = (s+1)\hat{\mathbf{i}} + (2s-1)\hat{\mathbf{j}} - (2s+1)\hat{\mathbf{k}}$$

16. If $y = \log\left|x + \sqrt{x^2+1}\right|$, then prove that $(x^2+1)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = 0$.

17. A kite is moving horizontally at a height of 151.5 m. If the speed of kite is 10 m/s, then how fast is the string being let out, when the kite is 250 m away from the boy who is flying the kite, if the height of boy is 1.5 m?

Section C

18. Evaluate $\int \frac{2x-1}{(x-1)(x+2)(x-3)} dx$.

19. Two schools, P and Q, want to award their selected students for the values of sincerity, truthfulness and hard work at the rate of ₹ x, ₹ y and ₹ z for each respective value per student. School P awards its 2, 3 and 4 students on the above respective values with a total prize money of ₹ 4600. School Q wants to awards its 3, 2 and 3 students on the respective values with a total award money of ₹ 4100. If the total amount of award money for one prize on each value is ₹1500, using matrices find the award money for each value. Suggest one other value which the school can consider, for awarding the students.

20. Let $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and R be a relation in $A \times A$, defined by $(a, b)\ R\ (c, d) \Leftrightarrow a + d = b + c$ for all (a, b) and $(c, d) \in A \times A$. Prove that R is an equivalence relation. Also, obtain the equivalence class determined by $(2, 5)$.

21. Find the area of the smaller region bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and the straight line $\frac{x}{a} + \frac{y}{b} = 1$.

or

Evaluate $\int_{-1}^{1} e^x\, dx$ as limit of sums.

22. Find the length and the equations of the line of shortest distance between the lines $\frac{x-8}{3} = \frac{y+9}{-16} = \frac{z-10}{7}$ and $\frac{x-15}{3} = \frac{y-29}{8} = \frac{z-5}{-5}$.

23. Find the particular solution of the differential equation $\frac{dx}{dy} + x\cot y = 2y + y^2 \cot y (y \neq 0)$, given that $x = 0$ when $y = \frac{\pi}{2}$.

or

Solve the differential equation $(1 + e^{2x})dy + e^x(1 + y^2)\, dx = 0$ it being given that $y = 1$ when $x = 0$.

24. Three critics review a book. For the three critics, the odds in favour of the book are (5 : 2), (4 : 3) and (3 : 4) respectively. Find the probability that the majority is in favour of the book.

25. A wire of length 36 cm is cut into two pieces. One of the pieces is turned in the form of a square and the other in the form of an equilateral triangle. Find the length of each piece, so that the sum of the areas of the two be minimum.

26. A chemical industry produces two compounds, A and B. The following table gives the units of ingredients C and D (per kg) of compound A and B as well as minimum requirements of C and D and costs per kg of A and B.

	Compound (in units)		Minimum requirement (in units)
	A	B	
Ingredient C (per kg)	1	2	80
Ingredient D (per kg)	3	1	75
Cost per kg (in ₹)	4	6	

Find the quantities of A and B which would minimise the cost.

Answers of Board Exam Corner (December Issue)

1. $\begin{vmatrix} a & b & c \\ a & b & c \\ x & y & z \end{vmatrix} + 2\begin{vmatrix} a & b & c \\ x & y & z \\ x & y & z \end{vmatrix} = 0$

2. $\text{adj}(AB) = \text{adj}(B) \cdot \text{adj}(A) = \begin{bmatrix} 2 & -3 \\ -5 & 2 \end{bmatrix}\begin{bmatrix} 3 & 5 \\ 7 & -2 \end{bmatrix} = \begin{bmatrix} -15 & 16 \\ -1 & -29 \end{bmatrix}$

3. We know that the range of principal value branch of $\cos^{-1}$ and $\sin^{-1}$ are $[0, \pi]$ and $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$, respectively.

The principal value of $\cos^{-1}\left(\cos\frac{2\pi}{3}\right) + \sin^{-1}\left(\sin\frac{2\pi}{3}\right)$

$= \cos^{-1}\left(\cos\frac{2\pi}{3}\right) + \sin^{-1}\left[\sin\left(\pi - \frac{\pi}{3}\right)\right] = \pi$

4. We have, $A^2 = A \cdot A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$; $A^4 = A^2 \cdot A^2 = \begin{bmatrix} 8 & 8 \\ 8 & 8 \end{bmatrix} = 8A$

$\therefore \quad \lambda = 8 \qquad [\because A^4 = \lambda A, \text{given}]$

5. On squaring the given differential equation, we get

$$\left(\frac{d^2y}{dx^2}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2 \qquad \ldots(i)$$

Hence, degree of Eq. (i) is 2.

6. We have, $\tan^{-1}\frac{3}{4} + \tan^{-1}\frac{3}{5} = \tan^{-1}\left[\frac{\frac{3}{4}+\frac{3}{5}}{1-\frac{3}{4}\times\frac{3}{5}}\right] = \tan^{-1}\left[\frac{27}{11}\right]$

$[\because \tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right), xy < 1]$

7. f is one-one. For any $x_1, x_2 \in N$ (domain), we have

$f(x_1) = f(x_2)\ x_1^2 + x_1 + 1 = x_2^2 + x_2 + 1$

$(x_1^2 - x_2^2) + (x_1 - x_2) = 0 \Rightarrow x_1 - x_2 = 0 \quad [\text{As } x_1 + x_2 + 1 \neq 0 \text{ for any } N]$

$\Rightarrow \quad x_1 = x_2, \forall\ x_1, x_2 \in N$

So, f is one-one, clearly $f(x) = x^2 + x + 1 \geq 3$ for $x \in N$.

But $f(x)$ does not assume values 1 and 2.

$f : N \to N$ is not onto function. **Hence proved.**

8. We have,

LHS $= \sin[\cot^{-1}\{\cos(\tan^{-1}x)\}]$

$= \sin[\cot^{-1}\{\cos A\}] = \sin\left[\cot^{-1}\frac{1}{\sqrt{1+x^2}}\right]$

$\left[\begin{matrix}\text{where, } \tan^{-1}x = A \text{ or } x = \tan A \\ \text{and} \sec^2 A = 1 + \tan^2 A\end{matrix}\right]$

$= \sin B \left[\text{where, } \cot^{-1}\frac{1}{\sqrt{1+x^2}} = B \text{ or } \cot B = \frac{1}{\sqrt{1+x^2}}\right]$

$= \frac{1}{\sqrt{1+\cot^2 B}} = \sqrt{\frac{x^2+1}{x^2+2}} = \text{RHS} \qquad \left[\because \cot B = \frac{1}{\sqrt{1+x^2}}\right]$

or

Let $\sin^{-1}\frac{3}{5} = x \text{ and } \sin^{-1}\frac{8}{17} = y$, then $\sin x = \frac{3}{5}$ and $\sin y = \frac{8}{17}$ ...(i)

Now, $\cos x = \sqrt{1-\sin^2 x} = \frac{4}{5}$

and $\cos y = \sqrt{1-\sin^2 y} = \frac{15}{17}$ ...(ii)

We know that, $\cos(x-y) = \cos x \cos y + \sin x \sin y$

$= \frac{60}{85} + \frac{24}{85} = \frac{84}{85}$ [using Eqs. (i) and (ii)]

Hence, $\sin^{-1}\frac{3}{5} - \sin^{-1}\frac{8}{17} = \cos^{-1}\frac{84}{85}$ **Hence proved.**

9. On applying $R_1 \to xR_1, R_2 \to yR_2, R_3 \to zR_3$ in LHS and dividing it by xyz

$$= \frac{1}{xyz}\begin{vmatrix} x(y+z)^2 & x^2y & x^2z \\ xy^2 & y(x+z)^2 & y^2z \\ xz^2 & yz^2 & z(x+y)^2 \end{vmatrix}$$

Taking common factors x, y, z from C_1, C_2 and C_3 respectively and then applying $C_1 \to C_1 - C_3$ and $C_2 \to C_2 - C_3$,

$$= \begin{vmatrix} (y+z)^2 - x^2 & 0 & x^2 \\ 0 & (x+z)^2 - y^2 & y^2 \\ z^2 - (x+y)^2 & z^2 - (x+y)^2 & (x+y)^2 \end{vmatrix}$$

Taking common factors $(x+y+z)$ from C_1 and C_2 and then applying $R_3 \to R_3 - R_1 - R_2$, $= (x+y+z)^2\begin{vmatrix} y+z-x & 0 & x^2 \\ 0 & x+z-y & y^2 \\ -2y & -2x & 2xy \end{vmatrix}$

Applying $C_1 \to C_1 + \frac{1}{x}C_3$ and $C_2 \to C_2 + \frac{1}{y}C_3$ and then along R_3,

$= 2xyz(x+y+z)^3$ **Hence proved.**

10. $y = \cos^{-1}\left[\frac{3x + 4\sqrt{1-x^2}}{5}\right] = \cos^{-1}\left[\frac{3}{5}\sin\theta + \frac{4}{5}\cos\theta\right]$

$[\because \text{put } x = \sin\theta, \text{ then } \theta = \sin^{-1}x]$

$= \cos^{-1}[\cos(\alpha - \theta)] \qquad [\because \text{where}, \cos\alpha = \frac{4}{5} \text{and} \sin\alpha = \frac{3}{5}]$

$= \alpha - \theta$

$y = \tan^{-1}\frac{3}{4} - \sin^{-1}x \quad [\text{where } \tan\alpha\frac{3}{4} \Rightarrow \alpha = \tan^{-1}\frac{3}{4}]$

$\frac{dy}{dx} = -\frac{1}{\sqrt{1-x^2}}$ [on differentiating]

or

By definition $|x-3| = \begin{cases} x-3, & \text{if } x \geq 3 \\ -(x-3) = -x+3, & \text{if } x < 3 \end{cases}$

For continuity at $x = 3$,

LHL $= \lim_{x \to 3^-} f(x) = \lim_{h \to 0}[-(3-h)+3] = 0$

RHL $= \lim_{x \to 3^+} f(x) = \lim_{h \to 0}[(3+h)-3] = 0$

Also, $f(3) = |3-3| = |0| = 0, \quad \lim_{x \to 3^-} f(x) = 0 = f(3) = \lim_{x \to 3^+} f(x)$

Therefore, f is continuous at $x = 3$. For differentiability at $x = 3$

LHD (at $x = 3$) $= Lf'(3) = \lim_{x \to 3^-}\frac{f(x) - f(3)}{x - 3} = -1$

RHD (at $x = 3$) $= Rf'(x) = \lim_{x \to 3^+}\frac{f(x) - f(3)}{x - 3} = 1$

Since, $Lf'(3) = -1 \neq 1 = Rf'(3)$, therefore $f(x)$ is not differentiable at $x = 3$

11. Taking log both sides in given equation, we get

$y \log\cos x = x \log\cos y$

Differentiating both sides w.r.t. x, we get $\frac{dy}{dx} = \frac{\log\cos y + y\tan x}{\log\cos x + x\tan y}$

or

Given, $x = \frac{\sin y}{\sin(a+y)}$ ...(i)

Differentiating Eq. (i) w.r.t. y, we get

$\frac{dx}{dy} = \frac{\sin[(a+y)-y]}{\sin^2(a+y)} \Rightarrow \frac{dy}{dx} = \frac{\sin^2(a+y)}{\sin a}$ **Hence proved.**

12. Using formula $= \frac{4}{3}\pi r^3$. Loss to the buyer = Error in volume × Cost

A small error of 0.04 cm can result in huge loss of ₹ 24640. So, we should careful while taking measurement of items.

13. We have, LHL $= \lim_{x \to 3^-} f(x) = \lim_{h \to 0}[a(3-h)+1] = 3a + 1$

and $RH\alpha = \lim_{x \to 3^+} f(x) = \lim_{h \to 0}[b(3+h)+3) = 3b + 3$

Also, $f(3) = a(3) + 1 = 3a + 1$

Since, f is continuous at $x = 3$, therefore $f(3) = \{LHL\} = RHL$

Hence, relation between 'a' and 'b' is $3a - 3b - 2 = 0$.

14. On taking common a, b, c from R_1, R_2, R_3 respectively and then from C_1, C_2, C_3 respectively $\Delta = a^2b^2c^2\begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$

Applying $C_1 \to C_1 + C_3$ and $C_2 \to C_2 - C_3$ and then expand along R_1, we get $\Delta = a^2b^2c^2[0 - 0 + 1(4+0)] = 4a^2b^2c^2$ **Hence proved.**

15. ***f* is one-one** For any $x_1, x_2 \in R$, we have $f(x_1) = f(x_2)$

$x_1^2 + 4 = x_2^2 + 4 \Rightarrow (x_1^2 - x_2^2) = 0 \Rightarrow (x_1 - x_2) = 0$

$\Rightarrow \quad x_1 = x_2 \qquad [\because x_1 + x_2 \neq 0]$

So, f is one-one.

f is onto Let $y \in]4, \infty[$, then $f(x) = y\ x^2 + 4 = y \ \Rightarrow\ x = \sqrt{y-4}$

Clearly, $x \in R$ for all $y \in [4, \infty[$

Thus, for each $y \in [4, \infty[$, there exists $x = \sqrt{y-4} \in R$, such that

$$f(x) = x^2 + 4 = (y-4) + 4 = y$$

So, f is onto. Thus, f is both one-one and onto consequently, it is invertible.

Now, $fof^{-1}(x) = x$ for $\in R$

$[f^{-1}(x)] = \sqrt{x-4}$ for all $x \in [4, \infty[$

Now, we define $f^{-1} : \{[4, \infty[\to R : f^{-1}(y)\} = \sqrt{y-4}$

for $x \in [4, \infty[$ **Hence proved.**

16. We have, $MC(x) = \text{Marginal Cost}\ \dfrac{dC}{dx} = 2x\sqrt{x+5}$

$C(x) = \text{Total Cost} = \int 2x\sqrt{x+5}\,dx + K = \dfrac{4}{15}(x+5)^{3/2}\,[3x-10] + K$ ...(i)

It is given that the cost of producing 4 units of the product is ₹ 314.40 when $x = 4, C(x) = 314.40$, $314.40 = \dfrac{4}{15}(9)^{3/2}\,[12-10] + K \Rightarrow K = 300$

Putting $K = 300$ in Eq. (i), we get $C(x) = \dfrac{4}{15}(x+5)^{3/2}\,(3x-10) + 300$

17. (i) Let x, y and z be the amount of investment at the rate of 6%, 8% and 9% per annum respectively, then

$$\begin{bmatrix} 1 & 1 & 1 \\ 0.06 & 0.08 & 0.09 \\ 0 & -0.08 & .09 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6500 \\ 4800 \\ 600 \end{bmatrix}$$

(ii) $|A| = \begin{vmatrix} 1 & 1 & 1 \\ 6 & 8 & 9 \\ 0 & -8 & 9 \end{vmatrix} = 42 \neq 0$

Therefore, the given system of equations can solve by using matrix method.

or

(i) Let x and y be the initial investment in 10% bond and 15% bond respectively. Then, according to the given statement, we have

$0.10x + 0.15y = 4000 \Rightarrow 2x + 3y = 80000$ ...(i)

$0.10\left(x + \dfrac{20}{100}x\right) + 0.15\left(y + \dfrac{10}{100}\right)y = 4000 + 500$

$8x + 11y = 300000$ ...(ii)

The linear Eqs. (i) and (ii) can be written in matrix form as $AX = B$, where, $A = \begin{bmatrix} 2 & 3 \\ 8 & 11 \end{bmatrix}$ $X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $B = \begin{bmatrix} 80000 \\ 300000 \end{bmatrix}$

(ii) $|A| = \begin{vmatrix} 2 & 3 \\ 8 & 11 \end{vmatrix} = 22 - 24 = -2 \neq 0$

Therefore, the given system of equations can solve by using matrix method.

18. Now, $|B| = \begin{vmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{vmatrix} = 1 \neq 0$

Since, $|B| \neq 0$, therefore B^{-1} exists $= \dfrac{1}{|B|}\text{adj}\ B = \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$ $[\because |B| = 1]$

We know that, $(AB)^{-1} = B^{-1}A^{-1}$

$\therefore \quad (AB)^{-1} = \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}\begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix} = \begin{bmatrix} 9 & -3 & 5 \\ -2 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}$

or

$$A^2 = A \cdot A = \begin{bmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{bmatrix};\ A^3 = A^2 \cdot A = \begin{bmatrix} 8 & 7 & 1 \\ -23 & 27 & -69 \\ 32 & -13 & 58 \end{bmatrix}$$

$$A^3 - 6A^2 + 5A + 11I = \begin{bmatrix} 8 & 7 & 1 \\ -23 & 27 & -69 \\ 32 & -13 & 58 \end{bmatrix} - 6\begin{bmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{bmatrix} + 5$$

$$+ 5\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix} + 11\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 0$$

$A^{-1}(A^3 - 6A^2 + 5A + 11I) = A^{-1}\ 0$ [pre-multiplying both sides by A^{-1}]

$11A^{-1} = -A^2 + 6A - 5I$

On solving this equation, we get $A^{-1} = \dfrac{1}{11}\begin{bmatrix} -3 & 4 & 5 \\ 9 & -1 & -4 \\ 5 & -3 & -1 \end{bmatrix}$

19. Let r, h and S be the radius, height and surface area of a closed right circular cylinder respectively, then

$S = 2\pi r^2 + 2\pi rh \ \Rightarrow h = \dfrac{S - 2\pi r^2}{2\pi r}$ [given] ...(i)

Volume of the right circular cylinder,

$V = \pi r^2 h;\ V = \dfrac{1}{2}[Sr - 2\pi r^3]$ [$\because$ from Eq. (i)] ...(ii)

$\dfrac{dV}{dr} = \dfrac{1}{2}[S - 6\pi r^2]$ [on differentiating] ...(iii)

For maximum or minimum, $\dfrac{dV}{dr} = 0$

From Eqs. (iii) and (i), we get $h = 2r$

$\therefore$ Height of closed cylinder = Diameter of a closed cylinder

Now, $\dfrac{d^2V}{dr^2} = \dfrac{d}{dr}\left[\dfrac{1}{2}\{S - 6\pi r^2\}\right] = -6\pi r < 0$

V is maximum when $h = 2r$

or

Let r, h and S be the radius, height and total surface area of an open right circular cylinder respectively. Then,

$S = \pi r^2 + 2\pi rh\ ;\ h = \dfrac{S - \pi r^2}{2\pi r}$ [given] ...(i)

Let V be the volume of the cylinder, then $V = \pi r^2 h$

$V = \dfrac{1}{2}[Sr - \pi r^3]$ [From Eqs. (i)]

Differentiating both sides w.r.t. r, we get

$\dfrac{dV}{dr} = \dfrac{1}{2}[S - 3\pi r^2]$ ...(ii)

For maximum of minimum, $\dfrac{dV}{dr} = 0$

Now, $\dfrac{d^2V}{dr^2} = \dfrac{d}{dr}\left[\dfrac{1}{2}(S - 3\pi r^2)\right] = -3\pi r < 0$

V is maximum when $h = r$.

20. Now, $|A| = \begin{vmatrix} 4 & -5 & -11 \\ 1 & -3 & 1 \\ 2 & 3 & -7 \end{vmatrix} = -72 \neq 0$

Since, $|A| \neq 0$, therefore A is invertible.

Thus, $A^{-1} = \dfrac{1}{|A|}\text{adj}\,(A) = \dfrac{1}{72}\begin{bmatrix} 18 & -68 & -38 \\ 9 & -6 & -15 \\ 9 & -22 & -7 \end{bmatrix}$

Given system of equations can be written in matrix form as

$AX = B$ i.e. $\begin{bmatrix} 4 & -5 & -11 \\ 1 & -3 & 1 \\ 2 & 3 & -7 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 12 \\ 1 \\ 2 \end{bmatrix}$

Now, $AX = B\ ;\ X = A^{-1}B$

On solving this, we get $x = -1$, $y = -1$ and $z = -1$

21. On differentiating given equation w.r.t. x_1, we get

$2x = 4\dfrac{dy}{dx} \ \Rightarrow \dfrac{dy}{dx} = \dfrac{x}{2} \Rightarrow \left[\dfrac{dy}{dx}\right]_{(h,\,k)} = \dfrac{h}{2}$

$\therefore$ Equation of normal at (h, k) and slope $= \dfrac{-2}{h}$ is $y - k = \dfrac{-2}{h}(x - h)$

Since, the normal passes through the point $(1, 2)$,

$2 - k = \dfrac{-2}{h}(1 - h) \Rightarrow k = 2 + \dfrac{2}{h}(1 - h)$ ...(i)

Also (h, k) lies on the curve $x^2 = 4y \Rightarrow h^2 = 4k$ ...(ii)

On solving Eqs. (i) and (ii), we get $h = 2$ and $k = 1$

From Eq. (i), we have $y - 1 = \dfrac{-2}{2}(x - 2) \Rightarrow x + y = 3$

Now, equation of tangent at (h, k) is $y - k = \dfrac{h}{2}(x - h)$

On putting $h = 2$ and $k = 1$, we get $y = x - 1$

22. Let r be the radius of base, h be the height at any stage and V be the volume of water in conical funnel.

Here, radius of base of the conical funnel is 5 cm and height of the conical funnel is 10 cm.

$\therefore \quad \frac{dV}{dt} = -5\,cm^3/s$ [–ve sign taken for leaking of water]

and $h = 10 - 2.5 = 7.5\ cm \Rightarrow r = h/2$

$\left[\Delta ABC \sim \Delta ADE, \frac{AB}{AD} = \frac{BC}{DE}\right]$

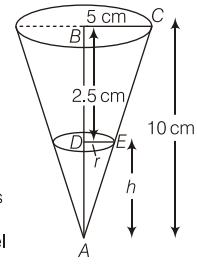
$\because$ Volume of cone $= \frac{1}{3}\pi r^2 h$

$\therefore$ Volume of water in conical funnel $(V) = \frac{\pi}{12}h^3$

$\frac{dV}{dt} = \frac{\pi}{4}h^2\frac{dh}{dt}$ [On differentiation]

$\Rightarrow \quad \frac{dh}{dt} = \frac{-20\times 100}{\pi \times 5625} = \frac{-16}{45\pi}\ cm/s$

Hence, the rate at which the water level dropping is $\frac{16}{45\pi}$ cm/s.



23. Let $u = (\tan^{-1}x)^y$ and $v = y^{\cot x}$ ...(i)

Then, given Eq. (i) becomes, $u + v = 1 \Rightarrow \frac{du}{dx} + \frac{dv}{dx} = 0$ ...(ii)

On taking log both sides in eqs. (i) and then differentiate

$\Rightarrow \quad \frac{du}{dx} = (\tan^{-1}x)^y\left[\frac{y}{(1+x^2)\tan^{-1}x} + \frac{dy}{dx}\log(\tan^{-1}x)\right]$

and $\frac{dv}{dx} = (y^{\cot x})\left[\frac{\cot x}{y}\frac{dy}{dx} - \text{cosec}^2\, x\log y\right]$

From Eq. (ii), we get

$$\frac{dy}{dx} = \frac{(y^{\cot x})(\text{cosec}^2\, x\log y) - (\tan^{-1}x)^y\left[\frac{y}{(1+x^2)\tan^{-1}x}\right]}{(\tan^{-1}x)^y\log\tan^{-1}x + y^{\cot x}\frac{\cot x}{y}}$$

24. On applying $C_1 \to C_1 - C_2$ and $C_2 \to C_2 - C_3$, we get

$$= \begin{vmatrix} (b-a)(a+b+c) & (c-b)(a+b+c) & ab - c^2 \\ (c-b)(a+b+c) & (a-c)(a+b+c) & bc - a^2 \\ (a-c)(a+b+c) & (b-a)(a+b+c) & ca - b^2 \end{vmatrix}$$

Taking $(a+b+c)$ common from each C_1 and C_2 and then apply, $R_1 \to R_1 + R_2 + R_3$, we get

$$\Delta = (a+b+c)^2\begin{vmatrix} 0 & 0 & ab+bc+ca-(a^2+b^2+c^2) \\ c-b & a-c & bc-a^2 \\ a-c & b-a & ca-b^2 \end{vmatrix}$$

Now, expanding along R_1, we get

$= \frac{1}{2}(a+b+c)(a^3+b^3+c^3-3abc)\ [(a-b)^2+(b-c)^2+(c-a)^2]$

Hence, given determinant is divisible by $(a+b+c)$ and quotient is

$(a^3+b^3+c^3-3abc)[(a-b)^2+(b-c)^2+(c-a)^2]$

25. LHL $= \lim\limits_{x\to 0^-} f(x) = \lim\limits_{h\to 0}\frac{e^{-\frac{1}{h}}-1}{e^{-\frac{1}{h}}+1} = -1$

and RHL $= \lim\limits_{x\to 0^+} f(x) = \lim\limits_{h\to 0}\frac{e^{\frac{1}{h}}-1}{e^{\frac{1}{h}}+1} = 1$

Here, LHL $\neq$ RHL

Hence, $f(x)$ is not continuous at $x = 0$.

26. Given geometric series are $\alpha = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$

$\therefore$ The sum of infinite geometric series $(\alpha) = \frac{a}{1-r} = \frac{2}{2-1} = 2$

and $\beta = 1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \ldots$

$\therefore$ The sum of infinite geometic series $(\beta) = \frac{a}{1-r} = \frac{1}{\frac{3-1}{3}} = \frac{3}{2}$

$\therefore \ \sec^{-1}\alpha + \sec^{-1}\beta = \sec^{-1}(2) + \sec^{-1}\left(\frac{3}{2}\right) = \cos^{-1}\left(\frac{1}{2}\right) + \cos^{-1}\left(\frac{2}{3}\right)$ ...(i)

Let $\cos^{-1}\left(\frac{1}{2}\right) = x \Rightarrow \cos = \left(\frac{1}{2}\right)$ and $\cos^{-1}\left(\frac{2}{3}\right) = y \ \Rightarrow \cos y = \frac{2}{3}$

Now, $\cos(x+y) = \cos x\cos y - \sin x\sin y$

$\Rightarrow \quad x + y = \cos^{-1}(\cos x\cos y - \sqrt{1-\cos^2 x}\sqrt{1-\cos^2 y})$

$\therefore$ Eq. (i) becomes $\sec^{-1}\alpha + \sec^{-1}\beta = \cos^{-1}\left[\frac{1}{3} - \frac{\sqrt{15}}{6}\right]$

Answers of Board Exam Corner (December Issue)

1. We have, $\mathbf{a}\cdot\mathbf{a} = 0$ and $\mathbf{a}\cdot\mathbf{b} = 0$, $\mathbf{a}\cdot\mathbf{a} = \mathbf{a}\cdot\mathbf{b}$

$\mathbf{a}\cdot\mathbf{a} - \mathbf{ab} = 0$, $\mathbf{a}(\mathbf{a}-\mathbf{b}) = 0$

Either $\mathbf{a} = 0$ or $\mathbf{a} - \mathbf{b} = 0$ or $\mathbf{a} \perp (\mathbf{a}-\mathbf{b})$

Either $\mathbf{a} = 0$ or $\mathbf{a} = \mathbf{b}$ or $\mathbf{a} \perp (\mathbf{a}-\mathbf{b})$

Thus, we conclude that $\mathbf{b} = \mathbf{a}$.

2. Given, $x\sqrt{1-y^2}\,dx + y\sqrt{1-x^2}\,dy = 0$; $\frac{dy}{dx} = \frac{-x\sqrt{1-y^2}}{y\sqrt{1-x^2}}$

Squaring both sides, we get $\left(\frac{dy}{dx}\right)^2 = \frac{x^2(1-y^2)}{y^2(1-x^2)}$

This differential equation is of first order and degree 2.

3. The direction cosines of a line are $\cos\alpha, \cos\beta, \cos\gamma$.

$\therefore \quad \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

$\Rightarrow \quad (1-\sin^2\alpha) + (1-\sin^2\beta) + (1-\sin^2\gamma) = 1$

$\Rightarrow \quad \sin^2\alpha + \sin^2\beta + \sin^2\gamma = 2$

4. Let $I = \int_e^{e^2}\frac{dx}{x\log x}$

Put $\log x = t \ \Rightarrow \ \frac{1}{x}dx = dt$

Also, when $x = e$, then $t = 1$ and when $x = e^2$, then $t = 2$

$\therefore \ I = \int_1^2\frac{dt}{t} = [\log t]_1^2 = \log 2 - \log 1 = \log 2$ [$\because \log 1 = 0$]

5. Given, $P(A) = \frac{2}{5}$, $P(B) = \frac{1}{3}$, $P(C) = \frac{1}{2}$; $P(A\cap C) = \frac{1}{5}$ and $P(B\cap C) = \frac{1}{4}$

$\because \ P\left(\frac{C}{B}\right) = \frac{P(B\cap C)}{P(B)} = \frac{1/4}{1/3} = \frac{3}{4}$ and $P(A'\cap C') = 1 - P(A\cup C)$

$= 1 - [P(A) + P(C) - P(A\cap C)]$

$= 1 - \left[\frac{2}{5} + \frac{1}{2} - \frac{1}{5}\right] = 1 - \left[\frac{4+5-2}{10}\right]$

$= 1 - \frac{7}{10} = \frac{3}{10}$

6. Let $I = \int\frac{e^{6\log x} - e^{5\log x}}{e^{4\log x} - e^{3\log x}}dx = \int\frac{e^{\log x^6} - e^{\log x^5}}{e^{\log x^4} - e^{\log x^3}}dx$

$= \int\frac{x^6 - x^5}{x^4 - x^3}dx = \int x^2dx = \frac{x^3}{3} + C$ [$\because e^{\log x} = x$]

7. Use formula $\int_0^a f(x)dx = \int_0^a f(a-x)dx = \frac{\pi^2}{16}$

or Use formula $\int_0^a f(x)dx = \int_0^a f(a-x)dx = \frac{\pi}{2}(\pi - 2)$

8. Convert the integral in proper rational function on dividing x^4 by $(x-1)(x^2+1)$.

$$\int\frac{x^4}{(x-1)(x^2+1)}dx = \frac{x^2}{2} + x + \frac{1}{2}\log|x-1| - \frac{1}{4}\log|x^2+1| - \frac{1}{2}\tan^{-1}x + C$$

9. Check the homogeneous nature of the differential equation, if it is homogeneous, then put $y = vx$ and

$$v + \frac{xdv}{dx} = \frac{dy}{dx};\ 1 + \log|x| - \cot\left(\frac{y}{x}\right) = 0$$

10. $P(X=1) = P$ (getting 1 defective bulb) $= \dfrac{{}^3C_1 \times {}^7C_2}{{}^{10}C_3}$

Similarly, for $P(x=2) = \dfrac{{}^3C_2 \times {}^7C_1}{{}^{10}C_3}$ and $P(x=3) = \dfrac{{}^3C_3 \times {}^7C_0}{{}^{10}C_3}$

x	0	1	2	3
P(x)	7/24	21/40	7/40	1/120

or Use formula $P(x) = {}^nC_r p^r q^{n-r}$ and $q = 1-p$

11. Use formula $\dfrac{x-x_1}{x_2-x_1} = \dfrac{y-y_1}{y_2-y_1} = \dfrac{z-z_1}{z_2-z_1} = \left(\dfrac{-5}{3}, \dfrac{2}{3}, \dfrac{19}{3}\right)$

12. Use $\left|\dfrac{(\mathbf{a}_2 - \mathbf{a}_1)\times \mathbf{b}}{|\mathbf{b}|}\right| = \dfrac{\sqrt{580}}{7}$ units

13. Put $x+y=z$, then $1+\dfrac{dy}{dx} = \dfrac{dz}{dx}$, $\log\left|1+\tan\dfrac{(x+y)}{2}\right| = x+C$

14. $n=3, p=\dfrac{1}{2}$, use $p(x=r) = {}^3C_r (p)^r (q)^{3-r}$ for $r = 0, 1, 2, 3 = \dfrac{3}{2}$

15. Use commutative law $= \dfrac{-3}{2}$

or

Use triangle law of vector addition to prove $\cos C = \dfrac{a^2+b^2-c^2}{2ab}$

16. Use one point form of plane

$a(x-x_1)+b(y-y_1)+c(z-z_1)=0, \quad x-19y-11z=0$

or

Use formula, $SD = \dfrac{\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{(b_1c_2-b_2c_1)^2+(c_1a_2-c_2a_1)^2+(a_1b_2-a_2b_1)^2}}$

$= 2\sqrt{29}$ units

17. Use formula, $\cos\theta = \dfrac{\mathbf{BA}\cdot\mathbf{BC}}{|\mathbf{BA}||\mathbf{BC}|} = \cos^{-1}\left(\dfrac{10}{\sqrt{102}}\right)$

The value of given message indicated 'Environmental protection

18. Use Baye's theorem 12/13.

19. Check the homogeneous nature of the differential equation, if it is homogeneous, then put $y=vx$ and $\dfrac{dy}{dx} = v + \dfrac{xdv}{dx} = 2xy\cos\dfrac{y}{x} = 3\pi$

20. It is in the form $\dfrac{dy}{dx} + py = Q$, where $p = 2\tan x$ and $Q = \sin x$.

$$y = \cos x - 2\cos^2 x$$

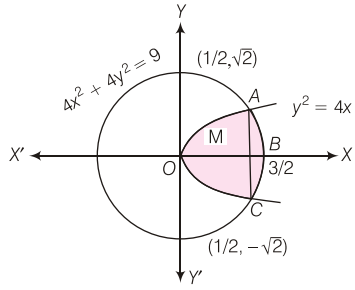
21. Required area of shaded region

$$= 2\left[\int_0^{1/2} y\,(\text{parabola})\,dx + \int_{1/2}^{3/2} y\,(\text{circle})dx\right]$$

$$\frac{\sqrt{2}}{6} + \frac{9\pi}{8} - \frac{9}{4}\sin^{-1}\left(\frac{1}{3}\right) \text{sq unit}$$

or

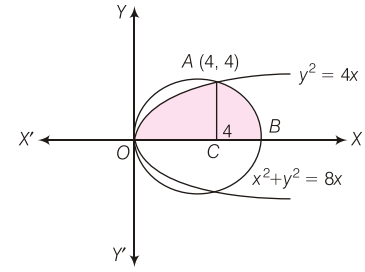
$\therefore$ Required area $= \int_{-1}^{1} y\,(\text{normal line } RP)dx + \int_{1}^{5+\sqrt{5}} y(\text{curve})dx$



$$= \frac{5640 - 10\sqrt{5}}{3} \text{ sq unit}$$

22. Required area of shaded region

$$= \int_0^4 y(\text{parabola})dx + \int_4^8 y(\text{circle})dx$$



$$= \left(\frac{32}{3} + 4\pi\right) \text{sq units}$$

23. Use formula, $\dfrac{x-x_1}{x_2-x_1} = \dfrac{y-y_1}{y_2-y_1} = \dfrac{z-z_1}{z_2-z_1} = \dfrac{2\sqrt{11}}{11}$ units

24. Given geometric series is 10,60, 360....

Here, first term $(a) = 10$ and common ratio $(r) = \dfrac{60}{10} = 6$

According to the given condition, $m = a \times 100$ and $n = r \times 100$

$\therefore\ m = 10\times 100$ and $n = 6\times 100 \Rightarrow m = 1000 \Rightarrow n = 600$

Let x passengers travel by executive class and y passengers travel by economy class. We construct the following table.

Class	Numbers of tickets	Profit (in ₹)
Executive	x	$1000x$
Economy	y	$600y$
Total	$x+y$	$1000x+600y$

So, our problem is to maximise $Z = 1000x + 600y$ ...(i)

Subject to the constraints are $x+y \le 200$ ...(ii)

$x \ge 20$ ... (iii)

$y - 4x \ge 0$ or $y \ge 4x$ ...(iv)

and $x \ge 0, y \ge 0$ ...(v)

Now, solve it by graphical method. Executive class = 40 tickets

Economy class = 160 tickets = ₹136000

or

Let refinery A runs for x days and refinery B runs for y days.

$\therefore$ The objective function is to minimise $Z = 400x + 300y$.

Subject to the constraints are $100x + 200y \ge 13000$

or $x + 2y \ge 130$ ...(i)

$300x + 400y \ge 20000$

or $3x + 4y \ge 200$ ...(ii)

$200x + 100y \ge 15000$

or $2x + y \ge 150$ ...(iii)

$\dfrac{100+300}{2}x + 100y \ge 15000$

If a, b and c are in arithmetic mean, then $b = \dfrac{a+c}{2}$, $x, y \ge 0$ ...(iv)

Now, solve it by graphical method $A = \dfrac{170}{3}$ days, $B = \dfrac{110}{3}$ days

25. The equation of the plane passing through the line of intersection of the given planes

$\mathbf{r}\cdot(\hat{\mathbf{i}}+2\hat{\mathbf{j}}+3\hat{\mathbf{k}}) - 4 = 0$ and $\mathbf{r}\cdot(2\hat{\mathbf{i}}+\hat{\mathbf{j}}-\hat{\mathbf{k}}) + 5 = 0$ is

$\mathbf{r}\cdot(1+2\hat{\mathbf{j}}+3\hat{\mathbf{k}}) - 4] + \lambda[\mathbf{r}\cdot(2\hat{\mathbf{i}}+\hat{\mathbf{j}}-\hat{\mathbf{k}}) + 5] = 0$

$\mathbf{r}[(1+2\lambda)\hat{\mathbf{i}} + (2+\lambda)\hat{\mathbf{j}} + (3-\lambda)\hat{\mathbf{k}} + (-4+5\lambda)] = 0$ ...(i)

If the plane (i) is perpendicular to the plane $\mathbf{r}\cdot(5\hat{\mathbf{i}}+3\hat{\mathbf{j}}-6\hat{\mathbf{k}}) + 8 = 0$

Then, $(1+2\lambda)(5) + (2+\lambda)(3) + (3-\lambda)(-6) = 0$

$\lambda = \dfrac{7}{19}$

Substituting this value of λ in Eq. (i), we get the required equation of the plane

$$\mathbf{r}\left[\left(1+\frac{14}{19}\right)\hat{\mathbf{i}} + \left(2+\frac{7}{19}\right)\hat{\mathbf{j}} + \left(3-\frac{7}{19}\right)\hat{\mathbf{k}} + \left(-4+\frac{35}{19}\right)\right] = 0$$

or $\mathbf{r}\cdot(33\hat{\mathbf{i}} + 45\hat{\mathbf{j}} + 50\hat{\mathbf{k}}) - 41 = 0$

26. Use formula, area of a parallelogram

$$= |\mathbf{p}\times\mathbf{q}| = \frac{1}{2}|\mathbf{a}\times\mathbf{b}| = \sqrt{\frac{62}{2}} \text{ sq unit}$$

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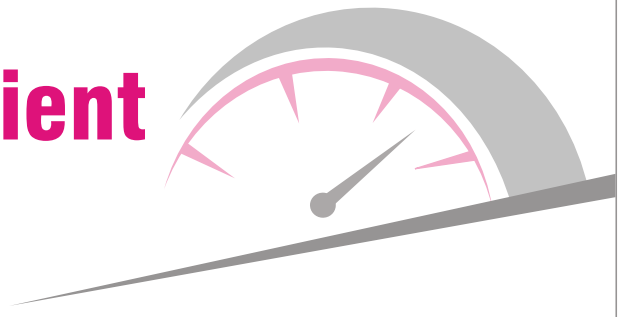
Knowledge Coefficient

Quizzer (No.15)



1. Let $f(x) = \int e^x (x-1)(x-2)\,dx$, then f decreases in the interval

(a) $(-\infty, -2)$ (b) $(-2, -1)$
(c) $(1, 2)$ (d) $(2, \infty)$

2. The equation of the circle of minimum radius which contains the three circles

$x^2 + y^2 - 4y - 5 = 0$

$x^2 + y^2 + 12x + 4y + 31 = 0$

and $x^2 + y^2 + 6x + 12y + 36 = 0$ is

(a) $\left(x + \frac{31}{18}\right)^2 + \left(y + \frac{23}{12}\right)^2 = \left(3 + \frac{5}{56}\sqrt{949}\right)^2$

(b) $\left(x - \frac{31}{18}\right)^2 + \left(y - \frac{23}{12}\right)^2 = \left(3 - \frac{5}{56}\sqrt{949}\right)^2$

(c) $\left(x + \frac{23}{12}\right)^2 + \left(y + \frac{31}{18}\right)^2 = \left(3 + \frac{5}{56}\sqrt{949}\right)^2$

(d) None of the above

3. Let $f(x) = ax^2 + bx + c, a, b, c \in R$ and $|f(x)| \le 1\ \forall\ |x| \le 1$, then the possible value of $|a + c|$, if $\frac{8}{3}a^2 + 2b^2$ is maximum, is given by

(a) 1 (b) 0
(c) 2 (d) 3

4. If the function $f(x) = \frac{t + 3x - x^2}{x - 4}$, where t is a parameter which has a minimum and a maximum value, then the range of value of t is

(a) (0, 4) (b) (0, ∞)
(c) (−∞, 4) (d) (4, ∞)

5. Let $f(x)$ be a function satisfying $f'(x) = f(x)$ with $f(0) = 1$ and g be the function satisfying $f(x) + g(x) = x^2$, then the value of the integral $\int_0^1 f(x) \cdot g(x)\,dx$ is

(a) $e - \frac{1}{2}e^2 - \frac{5}{2}$ (b) $e - e^2 - 3$
(c) $\frac{1}{2}(e - 3)$ (d) $e - \frac{1}{2}e^2 - \frac{3}{2}$

6. If the vectors $\mathbf{a} = \hat{\mathbf{i}} + a\hat{\mathbf{j}} + a^2\hat{\mathbf{k}}$, $\mathbf{b} = \hat{\mathbf{i}} + b\hat{\mathbf{j}} + b^2\hat{\mathbf{k}}$ and $\mathbf{c} = \hat{\mathbf{i}} + c\hat{\mathbf{j}} + c^2\hat{\mathbf{k}}$ are three non-coplanar vectors and $\begin{vmatrix} a & a^2 & 1 + a^3 \\ b & b^2 & 1 + b^3 \\ c & c^2 & 1 + c^3 \end{vmatrix} = 0$, then the value of abc is

(a) 0
(b) 1
(c) 2
(d) −1

7. If roots of the equation $z^2 + \alpha z + \beta = 0$ lie on $|z| = 1$ then

(a) $2\,|\text{Im}\,\alpha| = 1 - |\beta|^2$
(b) $2\,|\text{Im}\,\alpha| = |\beta|^2 - 1$
(c) $\text{Im}\,\alpha = 0$
(d) None of the above

8. If $(5 + 2\sqrt{6})^m = n + f$, where n and m are positive integers and $0 \le f < 1$, then $\frac{1 - f + f^2}{1 - f}$ is equal to

(a) $\frac{1}{n}$ (b) n
(c) $n + \frac{1}{n}$ (d) $n - \frac{1}{n}$

9. Let $\Delta = \begin{vmatrix} \sin^2 35^\circ & \sin^2 55^\circ & \tan(55^\circ + 80^\circ) \\ \sin^2 11^\circ & \tan(79^\circ + 66^\circ) & \sin^2 79^\circ \\ \cos^2 18^\circ & \cos^2 72^\circ & \cot(72^\circ + 63^\circ) \end{vmatrix}$,

then Δ is equal to

(a) -1 (b) 0

(c) 2 (d) $\frac{\sqrt{5}+1}{8}$

10. In a ΔABC, if $\cot A = (x^3 + x^2 + x)^{1/2}$, $\cot B = \left(x + \frac{1}{x} + 1\right)^{1/2}$ and $\cot C = \left(\frac{1}{x^3} + \frac{1}{x^2} + \frac{1}{x}\right)^{-1/2}$,

then the triangle is

(a) isosceles (b) obtuse angled

(c) right angled (d) equilateral

KNOWLEDGE Coefficient

Quizzer (No. 15)

Q. No.	a.	b.	c.	d.	Q. No.	a.	b.	c.	d.
1.	○	○	○	○	6.	○	○	○	○
2.	○	○	○	○	7.	○	○	○	○
3.	○	○	○	○	8.	○	○	○	○
4.	○	○	○	○	9.	○	○	○	○
5.	○	○	○	○	10.	○	○	○	○

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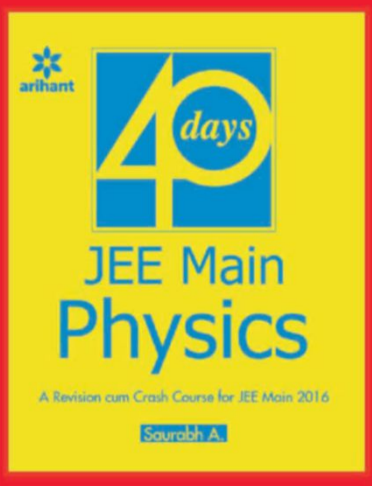

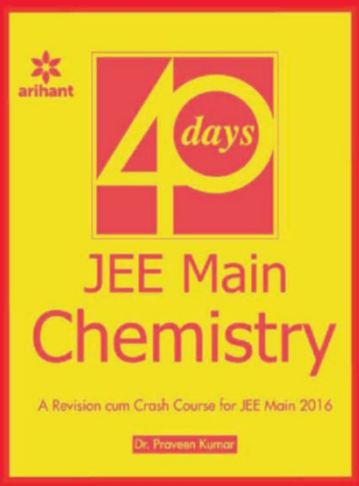

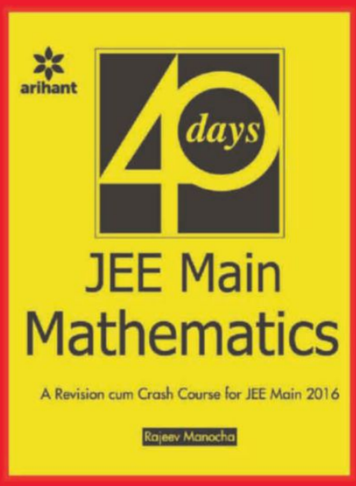

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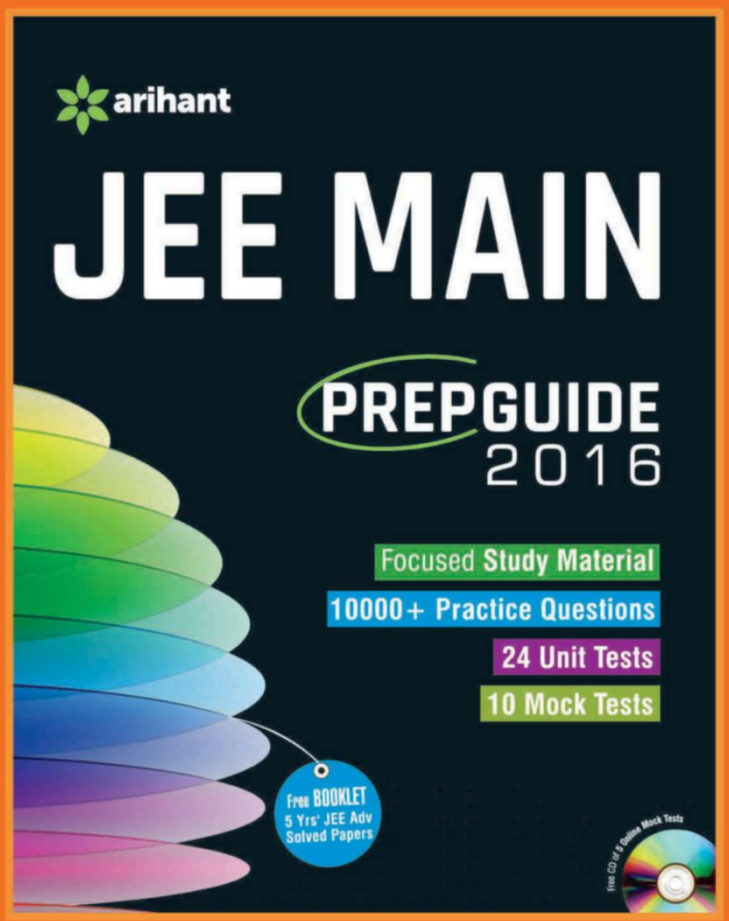

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